A SHORTER SCHOOL GEOMETRY

PART I.

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A SHORTER
SCHOOL GEOMETRY

PART I.

BY

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PREFACE.

OUR *School Geometry*, which has had a favourable reception for more than twenty years, was first issued at a time when the teaching of Geometry was in a transition state, passing from the strict formality of Euclid to methods of greater freedom and elasticity. In such circumstances the text-books of the day could only adopt a cautious and tentative treatment of their subject. Our own attitude was sufficiently indicated in the following paragraph quoted from our Preface, bearing date Nov. 1903:

"An attempt has been made to curtail the excessive body of text which the demands of Examinations have hitherto forced as 'bookwork' on a beginner's memory. Even of the Theorems here given a certain number (which we have distinguished with an asterisk) might be omitted or postponed at the discretion of the teacher. And the formal propositions for which—as such—teacher and pupil are held responsible, might perhaps be still further limited to those which make the landmarks of Elementary Geometry. Time so gained should be used in getting the pupil to *apply* his knowledge; and the working of examples should be made as important a part of a lesson in Geometry as it is so considered in Arithmetic and Algebra."

In the course of years fresh developments have arisen to clear the air in all branches of Elementary Mathematics. In Geometry there has been for some time a growing feeling in favour of simplifying the subject, by adopting a still shorter ordinary school course, and devising a sequence likely to be acceptable both to teachers and examiners. With these ends in view, in Jan. 1923 the Incorporated Association of Assistant Masters in Secondary Schools made a forcible

appeal to teachers throughout the kingdom, putting forward a Syllabus of Elementary Geometry on the lines above indicated. It was a genuine effort to reduce the confusion of the last twenty years to some sort of order and agreement, and embodied valuable recommendations as to the best use of time and material in the class-room.

Some radical revision of our *School Geometry* had long been overdue; but it is not easy to make changes satisfactorily in a text-book which is being widely used. It is inconvenient both to teachers and pupils if two or more editions are in use in the same class. Moreover, a mere revision—however thorough—would not have met the needs of the case, and would have entailed insuperable difficulties in a stereotyped book containing hundreds of diagrams.

Guided by such considerations, and with a distinct lead offered us by the Syllabus above mentioned, we have now undertaken the present work under the title *A Shorter School Geometry*. To a large extent it will be found to differ in plan and detail from our former work, though we have been able to draw much useful material from that source. Though not wholly in agreement with the new Syllabus, we have followed many of its suggestions, and have thereby effected substantial improvement in matters of order and arrangement, without sacrificing the distinguishing features which contributed so largely to the success of our earlier work.

The following special points may be mentioned:

(i) The *Introduction* (pages 1-42), which furnishes a full and systematic course of Experimental and Practical Geometry, to precede any theoretical work.

(ii) The very full treatment of *Congruence of Triangles*, both practically and theoretically, as given in the Introduction and in Section I. To emphasize this most important part of a beginner's course the examples given for practice are numerous and easy. See pages 30-38, 60-63, 68-73.

(iii) All through the book the examples have undergone careful revision and have been largely increased in number, many being supplied with diagrams and hints for solution. For the most part these have been distributed throughout the text in immediate connection with the propositions on which they depend. Other

exercises more miscellaneous in character, mainly derived from recent examination papers, will be found at the end of the different Sections.

(iv) In the presentment of the text marginal references to earlier propositions have been reduced to a minimum. Reasons for the different steps are usually given verbally; theorems are only occasionally quoted by number, and then with some special object.

(v) The diagrams, typographical details, and pagination have received very careful attention: allied propositions have been brought into close juxtaposition, while a theorem and its converse will usually be found facing each other so that both are under view at the same time. By suitable headings to sections and sub-sections we have made it easy for a teacher to find his way about the book. Finally, the pagination has been so adjusted that a reader will never find the argument broken by having to turn over a page.

The work in its finished form will probably be found sufficiently comprehensive for an ordinary school course, but it is not offered as a complete substitute for the *School Geometry* which, besides a fuller text, contains much additional matter adapted to the needs of those whose studies go beyond the usual school limits. For such readers a suitable sequel to the present work will be found in the *School Geometry*, Parts V. and VI., which can be obtained separately, or together in a single volume.

We have been unable to make use of the Report recently issued by the Mathematical Association, as it did not come into our hands till our own work was too far advanced. The whole of Part I. (the present volume) and much of Part II. was complete in manuscript, and some of it in the press, several months ago; but the progress of our work has been much delayed through the exigencies of old age and ill-health. It is hoped that the publication of Part II. (already far advanced) will not be subjected to similar interruptions.

We are indebted to several Examining Bodies for permission to use examples from their papers. In particular, our thanks are due to the courtesy of the Controller of His Majesty's Stationery Office, the Cambridge Syndicate for Local Examinations and Higher School Certificate Examinations, the Oxford and Cambridge Schools Exami-

nation Board, the University of London, the University of Bristol, the Joint Matriculation Board of the Northern Universities (Manchester, Liverpool, Leeds, Sheffield, and Birmingham), and the Central Welsh Board.

We wish also to express our thanks to Mr. H. C. Beaven of Clifton College, not only for valuable assistance in reading the proof-sheets, but also for many useful suggestions in all parts of the book.

H. S. HALL.
F. H. STEVENS.

April, 1924.

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PART I.

GEOMETRY.

INTRODUCTION.

DEFINITIONS AND FIRST PRINCIPLES ILLUSTRATED EXPERIMENTALLY.

I. AXIOMS.

ALL mathematical reasoning is founded on certain simple principles, the truth of which is so evident that they are accepted without proof. These self-evident truths are called **Axioms.**

For instance.

Things which are equal to the same thing are equal to one another.

Axioms are said to be *general* when, like that quoted above, they apply equally to magnitudes of all kinds. These do not need particular enumeration. Certain special axioms, relating only to *geometrical* magnitudes, will be stated from time to time as they are required.

II. POINTS, LINES, SURFACES.

Every beginner knows in a general way what is meant by a **point, a line,** and a **surface.** But in geometry these terms are used in a strict sense which needs some explanation.

1. A **point** has position, but is said to have *no magnitude.*

This means that we are to attach to a point no idea of size either as to *length* or *breadth,* but to think only where it is situated. A dot made with a sharp pencil may be taken as roughly representing a point; but small as such a dot may be, it still has *some* length and breadth, and is therefore not actually a geometrical point. The smaller the dot, however, the more nearly it represents a point.

2. A **line** has length, but is said to have *no breadth.*

A line is traced out by a moving point. If the point of a pencil is moved over a sheet of paper, the trace left represents a line. But such a trace, however finely drawn, has some degree of breadth, and is therefore not itself a true geometrical line. The finer the trace left by the moving pencil-point, the more nearly will it represent a line.

3. Proceeding in a similar manner from the idea of a line to the idea of a surface, we say that

A **surface** has length and breadth, but *no thickness.*

And finally,

A **solid** has length, breadth, and thickness.

Solids, surfaces, lines and points are thus related to one another:

(i) A solid is bounded by surfaces.

(ii) A surface is bounded by lines; and surfaces meet in lines.

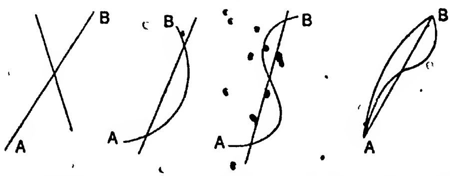
(iii) A line is bounded (or terminated) by points; and lines meet in points.

4. A line may be straight or curved.

A **straight** line has the same direction from point to point throughout its whole length.

A **curved** line changes its direction continually from point to point.

Several kinds of line (marked AB) are shown below, each having a *straight* line ruled across it for comparison.



Ex. 1. Mark a point on your paper, and call it A. How many straight lines, having different directions, can be drawn through the point A?

Ex. 2. Mark two points A and B. How many *straight* lines can be drawn through A and B? How many *curved* lines can be drawn from A to B?

Observe that the position and direction of a *straight* line are fixed if we know *two* points through which it passes.

Ex. 3. Can a straight line cut another *straight* line at more than one point? Can a straight line cut a *curved* line at more than one point?

Ex. 4. Of all the lines that can be drawn from a point A to a point B, which is the shortest?

Ex. 5. What is the least number of *straight* lines that can enclose a space? Can two *curved* lines enclose a space? Can *one* curved line enclose a space?

5. The above Exercises, which should in each case be illustrated by a drawing, lead to the following Axioms:—

Axioms. (i) *Two straight lines can cut at only one point.*

(ii) *Only one straight line can be drawn through two given points.*

(iii) *The straight line joining two points is the shortest distance between them.*

Note. When we rule a straight line from a point A to a point B we are said to join AB.

A finite (or terminated) straight line is said to be **produced** when it is prolonged to any length in that straight line.

To **bisect** means to divide into two *equal* parts.

In every finite straight line there is one point at equal distances from the two ends.

That is to say:

Every finite straight line has a point of bisection.

(Measurement of Straight Lines.)

6. For measuring straight lines the pupil will need a scale showing *inches and tenths of an inch* along one edge, and along the other edge *centimetres and millimetres.*

The few instructions required for the use of such a scale are best given orally.

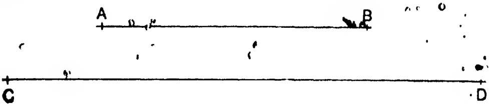
The units on the scale are divided into *tenths* in order that measurements may be recorded *decimally*: Thus

(i) *Three and seven-tenths inches* should be written 3·7 in., or 3·7″.

(ii) *Eight-tenths of an inch* should be written 0·8 in., or 0·8″.

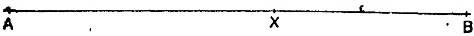
(iii) *Five centimetres four millimetres* should be written 5·4 cm.

Ex. 6. Measure the lengths of AB and CD in inches and tenths of an inch.

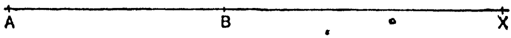


Ex. 7. Measure the above lines AB and CD as nearly as you can in centimetres and millimetres.

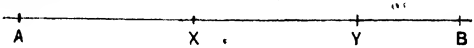
Ex. 8. Measure AX and XB in inches and tenths of an inch, and add your results together. Test your work by measuring AB.



Ex. 9. Measure AX and XB in centimetres and millimetres, and find their difference. Test your result by measuring AB.



Ex. 10. (i) Measure AB, AX, and XY in inches and tenths of an inch: hence reckon the length of YB, and test your result by measurement.



(ii) Measure AY, YB, and XB in centimetres, and hence find AY + YB − XB. What line should you now measure to test your result?

Ex. 11. Draw straight lines to show the following lengths:

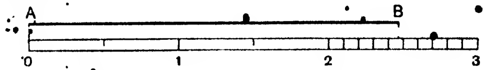
2·6 in.,	5·0 cm.,	1·8″,	4·7 cm.,	0·8 in.
8·2 cm.,	3·1″,	0·7 cm.,	9 mm.,	33 mm

Ex. 12. Draw a line AB of length 7·2 cm. By measurement cut off from it AP equal to *half* AB, and AQ equal to *one-third* AB. Find with your dividers how many times PQ is contained in AB. Explain your result by finding the value of $\frac{1}{2} - \frac{1}{3}$.

Ex. 13. Draw a line 4 inches in length. Measure it in terms of centimetres and millimetres. Divide your result by 4, and thus find the equivalent of 1 inch in cms.

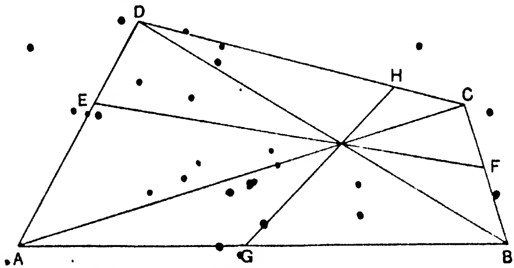
Ex. 14. Draw a line 10 centimetres in length. Measure it against your inch scale. Divide the result by 10, and thus find the equivalent of 1 centimetre in terms of inches.

7. So far lines measured in inches and tenths of an inch have contained an exact number of tenths. This will not always be so. For example



the line AB is more than 2·4″ and less than 2·5″. In this case we may mentally divide the tenth in which B falls into *ten* equal parts, that is to say, into *hundredths of an inch*, and judge as nearly as we can how many of these hundredths are to be added to 2·4. In this instance about *seven*-hundredths should be added, so that the length of AB is about 2·47″.

Ex. 15. Measure the straight lines named below; first in *inches, tenths*, and (as nearly as you can) in *hundredths*; then in *centimetres* and *millimetres*. Tabulate your results as indicated below

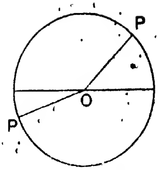


	inches	centimetres		inches	centimetres
AB =		=	EF =		=
BC =		=	AC =		=
CD =		=	BD =		=
DA =		=	GH =		=

8. A **plane** is a *flat* surface, the test of flatness being that if *any* two points are taken in the surface, the straight line between them lies wholly in that surface.

This may be illustrated by laying the straight-edge of a ruler on a table, and noticing that the *whole length* of the edge always rests upon the surface, *in whatever position the ruler is placed*. But if the ruler is placed in the hollow of a basin, only the ends rest on the surface.

III. Circles.



1. When a point (P) moves so that its distance from a fixed point (O) is always the same, the curved line so traced out is called the **circumference** of a **circle**, and the fixed point is called the **centre**.

NOTE. Strictly speaking the word *circle* means the figure enclosed by the curve, but it is often used as equivalent to the curve itself, namely the circumference.

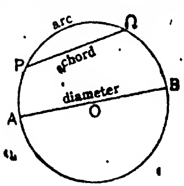
A circle is drawn with *compasses*, the steel point being kept at the centre, while the pencil point traces out the circumference. Every point on the curve so traced is at the same distance from the centre, for the span of the compasses is unaltered as they turn.

2. A straight line drawn from the centre of a circle to its circumference is called a **radius**. All radii of a circle are equal.

3. All circles are of the *same shape*, however large or small they may be.

Two circles of equal radius are of the same *shape* and *size*, and each is an exact copy of the other.

4. Any straight line drawn through the centre of a circle and terminated each way by the circumference is called a **diameter**.



An **arc** is any part of the circumference of a circle.

A **chord** is the straight line joining the ends of an arc; that is, the straight line joining any two points on the circumference.

EXERCISES ON CIRCLES.

Ex. 1. Take a distance of 5 centimetres in your compasses, and with a fixed point O as centre draw a circle. Note that (i) every point on the circumference must be 5 cm. distant from the centre O; (ii) every point which is 5 cm. from O must be on the circumference.

Explain why the circumference must be a *closed* curve.

Ex. 2. What is the greatest number of points at which a straight line can cut the circumference of a circle?

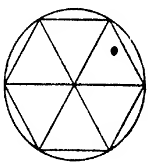
Ex. 3. Take a point O as centre and draw a circle with a radius, say, of 1·5 inches. Draw *any* diameter AB. Now carefully cut your circle out, and fold it about the diameter AB, thus dividing the circle into two parts. Do you find that one part fits exactly over the other? If so, this shows that the two parts *are of the same size and shape.*

The two equal parts into which a circle is divided by a diameter are called **semi-circles.**

Ex. 4. Draw a circle of diameter 3·0″, and on the circumference mark a point X. From X draw two chords, one 1·5″ long, the other 2·0″ long. What is the length of the longest chord in this circle?

Ex. 5. In the figure of Art. 4 notice that the chord PQ divides the circumference into *two* arcs. Point them out. Can a chord ever cut off two *equal* arcs? Which is the longer line, an arc, or the chord which joins its ends?

Ex. 6. Draw a circle, say of radius 2·0″, and with *the same radius* mark off points round the circumference. How many steps can you thus take? Six exactly. Are the *arcs* which you thus cut off each 2″ in length? Are they more or less than 2″? Join the points of division in order. Are the *chords* each 2″ in length? Why so? Join the centre to each point of division, and thus complete the pattern shown in the margin.



(Two or more circles. Intersection of circles.)

Ex. 7. Mark a point O on your paper, and from O as centre draw three circles, one of radius 3·5 cm., the next of radius 4·0 cm., the third of radius 4·5 cm. Notice that the circumferences do not cross or cut one another. Why not?

Circles which have the same centre are said to be concentric.

Ex. 8. (i) Take two points A and B, 7 cm. apart. With A as centre draw a circle of radius 4 cm.; and with B as centre draw a circle of radius 2 cm. Explain why each circle is outside the other. What is the shortest distance between the circumferences ?

(ii) Again take two points A and B, 7 cm. apart; and, as before, with A as centre draw a circle of radius 4 cm. But this time draw from centre B a circle of radius 5 cm. Why do these circles overlap ? At how many points do the circumferences cut one another ?

(iii) Once more take two points A and B, 7 cm. apart, and with A and B as centres draw two circles, one of radius 4 cm., the other of radius 3 cm. Do the circumferences cross one another ? Do they meet ? If your work is carefully done, the two circles just *touch* one another. Where is the touching point ? Say why.

Ex. 9 Take two points A and B, 2 cm. apart; and with centre A draw a circle of radius 5 cm. With centre B draw a circle of radius 3 cm. How does this circle meet the first, and where is the meeting-point ?

Ex. 10. Can you draw two circles which cut one another at more than two points ? Try.

Ex. 11. Take two points 3" apart, and call them A and B. With centre A and radius 2½" draw a circle. With centre B and radius 2" draw a second circle. Call the points at which the circles cut one another P and Q. How far is P from A and from B ? How far is Q from A and from B ?

Ex. 12. Take two points A and B, 3 cm. apart. Find with your compasses a point which is 6 cm. from A and also 6 cm. from B. Can you find more than one such point ? How many ?

Ex. 13. Draw a line 2·5" long, and find with your compasses a point that is 2·0" from each end. How many such points are there ?

Ex. 14. Take two points X and Y, 9 cm. apart. Find a point which is 6 cm. from X and 5 cm. from Y. How many such points are there ?

Ex. 15. Draw a line 3·3" long, and find two points each of which is 2·2" from one end and 1·8" from the other.

Ex. 16. Two forts defending the mouth of a river, one on each side, are 20 kilometres apart: their guns have an effective range of 12,000 metres. Draw a plan (scale 2 km. to 1 cm.) showing what part of the river is exposed to fire from both forts.

IV. Angles.

1. When two straight lines meet at a point, they are said to form an **angle**.

The straight lines are called the **arms** of the angle; and the point at which they meet is its **vertex**. The sign $\angle$ is used for the word *angle*.

The angle formed by the straight lines OA, OB is named *the angle* AOB, or BOA; if there is only one angle at a vertex, it may be named by a single letter, as *the angle* O.

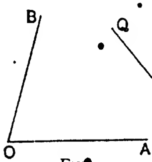


Fig. 1.

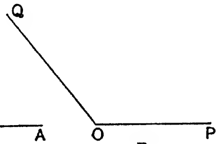


Fig. 2.

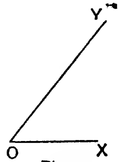


Fig. 3.

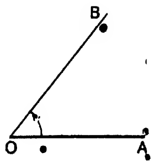
2. Figs. 1, 2, and 3 represent three angles. In Fig. 2 it is seen that the arms are *more widely opened out* than in Fig. 1; while in Fig. 3 the arms are *less widely opened out* than in Fig. 1. This we express by saying that

the angle POQ is *greater than* the angle AOB;
the angle XOY is *less than* the angle AOB.

Thus the size of an angle does not depend on the length of its arms, but only on the *slope* or *inclination* of one arm to the other.

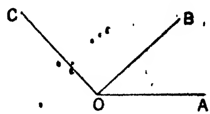
3. The magnitude of the angle may be thus explained:

Suppose that the arm OA is fixed, and that OB *turns about the point* O (as shown by the arrow). Suppose also that OB began its turning from the position OA. Then the size of the angle AOB is measured by the *amount of turning* required to bring the revolving arm from its first position OA into its subsequent position OB.



4. It is important to observe that two straight lines such as OA, OB, drawn from (or to) a point O, have *different directions*; and the angle thus formed measures the *difference* of direction.

5. Angles which lie on either side of a common arm are said to be **adjacent.**

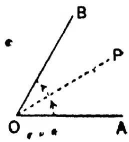


For example, the angles AOB, BOC, which have the common arm OB, are adjacent.

Observe that the two adjacent angles AOB, BOC together make up the whole angle AOC;

that is, the $\angle$AOB + the $\angle$BOC = the $\angle$AOC.

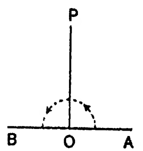
6. AXIOMS. (i) *If a line* OP, *revolving about* O, *turns from* OA *to* OB, *it must pass through one position (and only one) in which it divides the angle* AOB *into two equal parts.*



That is to say:

Every angle may be supposed to have a line of bisection.

(ii) *If* O *is a point in a straight line* AB, *then a line* OP *which turns about* O *from the position* OA *to the position* OB *must pass through one position (and one only) in which it makes the adjacent angles* AOP, POB *equal to one another.*



7. When one straight line stands on another so as to make the adjacent angles equal to one another, each of the angles is called a **right angle**; and each line is said to be **perpendicular** to the other.

AXIOMS. (i) *At a given point* O *in a straight line* AB *there is one, and only one, line perpendicular to* AB.

(ii) *All right angles are equal.*

Thus a right angle may be taken as a standard with which to compare other angles.

8. A right angle is divided into 90 equal parts called degrees (°).

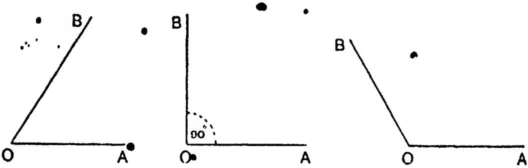
That is, *one right angle* $= 90^\circ$.

An **acute** angle is less than one right angle.

That is, an acute angle is less than 90°.

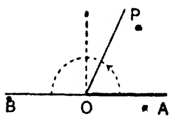
An **obtuse** angle is greater than *one* right angle, but less than *two* right angles.

That is, an obtuse angle lies between 90° and 180°.

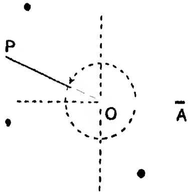


acute angle right angle obtuse angle

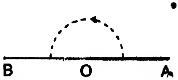
9. If O is a point in the straight line AB, and if OP revolves about O from the position OA into the position OB (as indicated in the figure), it turns through *two right angles*, or 180°.



If OP makes a *complete revolution* about O, starting from OA and returning to its original position (as indicated in the figure), it turns through *four right angles*, or 360°.

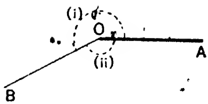


If one arm of an angle, starting from the position OA, turns about O until it makes a *straight line* with OA, the angle so formed is called a **straight** angle.



Thus the *straight angle* AOB $= 2$ right angles $= 180^\circ$.

10. An angle which is greater than *two* right angles, but less than *four* right angles, is said to be **reflex**.

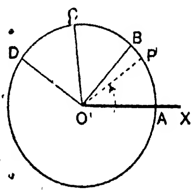


That is, a reflex angle lies between 180° and 360°.

NOTE. When two straight lines meet, *two* angles are formed, one greater, and one less than two right angles. The first arises by supposing OB to have revolved from the position OA the longer way round, marked (i); the other by supposing OB to have revolved the shorter way round, marked (ii). Unless the contrary is stated, the angle between two straight lines will be considered to be that which is less than two right angles.

(Angles at the Centre of a Circle.)

11. If a straight line OP of fixed length, revolving about O, makes a complete revolution (that is, turns through *four* right angles), the point P traces out the whole circumference of a circle; and while OP turns through any angle AOB, the point P traces out the corresponding arc AB.



Now suppose OP turns in succession through any *equal* angles, such as AOB, BOC, COD, we see at once that the corresponding *arcs* AB, BC, CD, traced out by the point P, must also be equal.

It follows that whatever part or parts the angle AOB is of *four right angles* (namely, a complete angular revolution), the same part or parts is the arc AB of the whole circumference.

For instance, if the ∠AOB = 45°, that is, *one-eighth* of 360°, then the arc AB = *one-eighth* of the whole circumference.

If the ∠AOD = 135°, that is, *three-eighths* of 360°, then the arc AD = *three-eighths* of the whole circumference. And so on.

We may also conclude that if the arcs AB, BC, CD are equal, then the angles at the centre opposite to them, namely, the ∠ˢ AOB, BOC, COD are also equal.

Ex. 1. What fractions of a *right angle* are 45°, 30°, 60°, $22\frac{1}{2}$°, $7\frac{1}{2}$°, $37\frac{1}{2}$° ?

Ex. 2. What fractions of a *straight angle* are 120°, 135°, 30°, $67\frac{1}{2}$°, $52\frac{1}{2}$°, 117° ?

Ex. 3. How many degrees are there in $\frac{1}{6}$, $\frac{1}{8}$, $\frac{2}{5}$, $\frac{5}{9}$, $\frac{15}{18}$ of a *complete angular revolution* ?

Ex. 4. On a clock-face what fractions of the whole circumference are arcs containing 10, 12, 36, 54, $37\frac{1}{2}$ minute divisions ?

Ex. 5. Through how many degrees does the minute-hand of a clock revolve in 1 hour, in $\frac{1}{4}$ hour, in $\frac{1}{2}$ hour, in $\frac{3}{4}$ hour ?

Ex. 6. Through how many degrees does the minute-hand revolve in 5 minutes, in 25 minutes, in 36 minutes ?

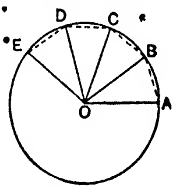
Ex. 7. How long will it take the minute-hand to turn through 48°, through 102°, through 9° ?

Ex. 8. If a wheel makes 10 revolutions a minute, through how many degrees will it turn in 1 second ?

Ex. 9. If the spoke of a wheel turns through an angle of 168° in 4 seconds, how many revolutions is the wheel making a minute ?

Ex. 10. The Earth makes a complete revolution about its axis in 24 hours. Through what angle will it turn in 2 hrs. 48 min. ? And how long will it take to turn through 225° ?

12. In the adjoining figure the points A, B, C, etc., are found by taking any distance on the compasses, and with it marking off equal distances round the circumference. The *lengths* thus 'stepped off,' represented by the dotted lines AB, BC, CD, etc, are all equal.

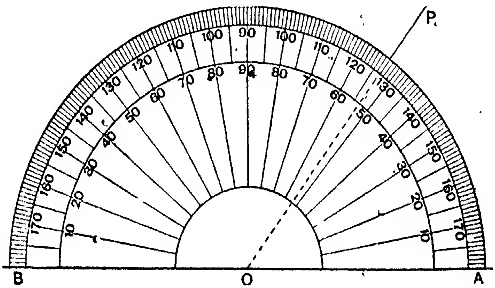


We have thus drawn a series of equal *chords*. The *arcs* cut off by these chords are also equal. Formal proof of this fact will be given at a later stage. Here its truth can only be tested experimentally. This may be done by cutting out, or folding, or by means of tracing.

It may now be assumed that in a circle (or in equal circles), when we step off equal chords, we thereby cut off equal arcs; and by cutting off equal arcs we can make equal angles at the centre.

Measurement of Angles. The Protractor.)

13. The principle illustrated on page 12 (namely, that *equal angles at the centre of a circle have equal arcs opposite to them*) is applied to the construction and use of the semi-circular **protractor** for the measurement of angles.



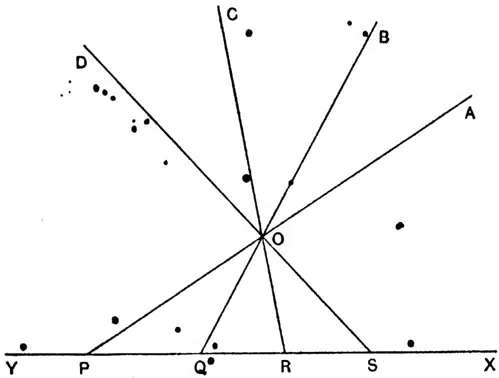
Your protractor shows a semi-circular arc divided into 180 equal parts, which for convenience are numbered from each end.

(i) *To measure the number of degrees in a given angle*, place the protractor with its centre at the vertex, and the diameter in line with one of the arms of the angle; then observe the mark of division on the rim under which the other arm passes.

(ii) *To make an angle of a given number of degrees* (*say* 53°), draw one arm OA; place the protractor with its centre on O and its diameter in line with OA; mark a point on your paper as close as you can to the 53rd division on the rim; remove the protractor and join the vertex O to the point so marked.

NOTE. In measuring a given angle, the pupil must be careful to read the number of degrees from the right set of numbers. Mistakes, however, can always be avoided by first observing whether the angle to be measured is acute or obtuse.

Ex. 11. In the following figure measure the angles named below:



$\angle$APX = ,	$\angle$BQX = ,	$\angle$CRX = ,	$\angle$DSX =
$\angle$APY = ,	$\angle$BQY = ,	$\angle$CRY = ,	$\angle$DSY =
$\angle$AOB = ,	$\angle$AOC = ,	$\angle$AOD = ,	$\angle$BOD =

Also measure the $\angle$s BOC, COP, POR. How many degrees are there in the *reflex* angle BOR? How can you determine this from the figure by a single measurement?

Ex. 12. Draw a straight line AB of length 3 inches. By means of your protractor draw lines making the following angles with AB:

(i) 62°, (ii) 27°, (iii) 81°, (iv) 157°, (v) 123°, (vi) 49°.

(This may be done in a single figure.)

Also from B draw lines making the above angles with BA.

Other angles with arms of convenient length for measurement should here be provided by the teacher.

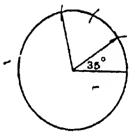
Ex. 13. Draw a straight line AB of length 8 cm. From A draw two lines, one on each side of AB, each making an angle of 47° with it. Repeat the process, making angles of 75° and 131° on each side of AB. (This is to be done in a single figure.)

If your figure were folded about AB, how would the lines on one side of AB fall with regard to those on the other side?

Ex. 14. Draw a line BC of length 3 inches. At B make an angle CBP equal to 32°, and at C make an angle BCQ equal to 53° on the same side of BC. If BP and CQ intersect at A, measure the four angles at that point.

Ex. 15. AD is a line 1·8 inches in length, and it is produced to B making DB equal to AD. At A make the angle BAC equal to 34°, and at B make the angle ABC equal to 56°. If the arms of the angles meet at C, and CD is joined, measure CD, and all the angles at C and D.

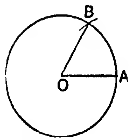
Ex. 16. Draw an angle of 35° with your protractor; then, with ruler and compasses, construct another angle *three times* the size of the first. Test your construction by measurement.



Ex. 17. I want to draw an angle *five times* as great as a given angle A. Explain in your own words how this may be done.

Ex. 18. Draw a circle with centre O and *any* radius. Step off this radius from A to B on the circumference, and join OA, OB.

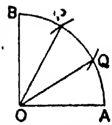
What fraction is the ∠AOB of *four right angles*, and why? How many degrees are there in the ∠AOB? Answer, then test by measurement.



Ex. 19. Draw an angle of 120°, using ruler and compasses only.

Ex. 20. With your protractor draw a *right angle* AOB. With centre O and any radius (say 7 cm.) draw the arc AB. What part of the whole circumference is this arc?

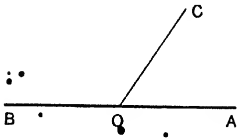
From centre A, with *the same radius*, cut the arc at P; and from centre B, with the same radius, cut the arc at Q. Join OP, OQ.



How large are the ∠s AOQ, QOP, POB? Answer, giving your reasons: then measure.

(*Adjacent Angles.*)

14. Draw a straight line AB, and from any point O in it draw another straight line OC in *any* direction.



Measure the angle AOC; and, *without moving the protractor*, measure the adjacent angle BOC. Add together the two results.

Evidently $\angle AOC + \angle BOC = 180$ degrees,

for whatever may be the direction of OC, the adjacent angles AOC, COB together make up the *straight* angle AOB: and we have seen that a straight angle is equivalent to 2 *right angles*, or 180°.

When the sum of two angles is equal to two right angles, each is said to be the **supplement** of the other.

Thus, in the above figure, the supplement of the $\angle AOC$ is the $\angle COB$. The supplement of 78° is 102°, for the sum of these angles = 180°. The supplement of x degrees is $180 - x$ degrees.

Ex. 21. As in the last figure OC is drawn from a point O in the straight line AB.

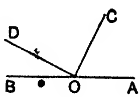
(i) if the $\angle AOC = 65°$, reckon the $\angle BOC$.

(ii) if the $\angle BOC = 140°$, reckon the $\angle AOC$.

(iii) if the $\angle AOC = 153°$, reckon the $\angle BOC$.

Ex. 22. Write down the supplements of the following angles: 45°, 60°, 120°, 71°, 135°, 148°; also of *one-third* of a right angle, *one-fifth* of a right angle, *two-thirds* of a straight angle, *three-quarters* of a straight angle.

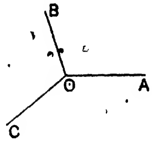
Ex. 23. Draw a straight line AB, and from a point O in it draw *any* straight lines OC, OD, on the same side of AB. Measure the angles AOC, COD, DOB, and find their *sum*. *Account for the result.*



Ex. 24. From a point O draw *any* three straight lines OA, OB, OC. Measure the ∠AOB, BOC, COA, and fill up the following:

∠AOB + ∠BOC + ∠COA = degrees.
= right angles.

Account for the result.

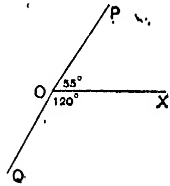


Ex. 25. In the figure of Ex. 24:

(i) If ∠AOB = 125°, and ∠BOC = 82°, reckon the ∠COA.
(ii) If ∠AOB = 134°, and ∠AOC = 152°, reckon the ∠BOC.

In each case test by measurement.

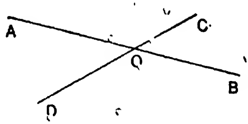
Ex. 26. Draw a straight line OX. Make the angle XOP = 55°; and on the other side of OX make the angle XOQ = 120°. Are OP and OQ in one straight line? If not, how should OQ be turned, so as to bring it into line with OP?



Ex. 27. Repeat the last exercise, making the angle XOP = 107°, and the angle XOQ = 66°. Through how many degrees must OQ be turned so as to bring it into line with OP? *State in words the principle which you are applying.*

(Vertically Opposite Angles.)

15. When two straight lines, such as AB, CD cross one another at O, the angles AOC, BOD are said to be **vertically opposite.** The angles AOD, COB are also vertically opposite to one another.



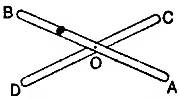
Draw straight lines AB, CD crossing at O at *any* angle.

Measure the angle AOC. Then observing that AOB is a *raight angle*, write down (without measurement) the number degrees in the ∠ COB.

Again, having measured the ∠ COA, and observing that OD is a straight angle, write down the number of degrees in e ∠ AOD.

It will be seen that the ∠ COB = the ∠ AOD; the reason ing that *each* of these vertically opposite angles is the *pplement* of the ∠ AOC.

Similarly the ∠ AOC = the vertically opposite ∠ BOD.



NOTE. The equality of vertically opposite gles may be illustrated by experiment.

For instance: two narrow strips of card-ard may be pivoted by a drawing-pin at Bring the strips into coincidence, then owly open them out. Observe that the me movement which opens the angle AOC, also opens the angle BOD: at is to say, these angles are the result of the *same amount of turning*, d are therefore equal to one another

Ex. 28. In the figure of Art. 15:

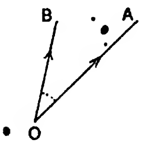
(i) If the ∠ BOD = 143°, reckon each of the ∠s BOC, COA, AOD.

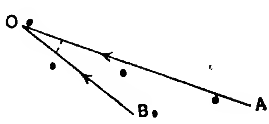
(ii) If the ∠ AOD = 29°, reckon each of the ∠s DOB, BOC, COA.

(iii) If the ∠ COA = 137°, reckon each of the ∠s BOD, DOA, COB.

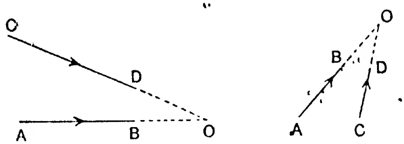
V. DIRECTION. PARALLELS.

1. We have seen (p. 10, Art. 4) that any two straight lines awn *from*, or *to*, a point O have *different directions*, and that e angle between them indicates the amount of this difference. other words, *the direction of* OB, *as compared with* OA, *is ed by the angle* AOB. [See figure below.]

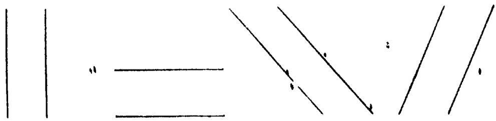




So also two straight lines (such as AB, CD in the figure below) which have different directions, will meet when produced at some point O, and form an angle, which measures the difference of direction.

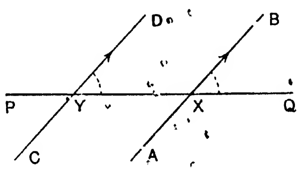


But two straight lines which have *like directions will never meet*, however far they are produced. Instances of straight lines which have like directions are shown below.



2. Straight lines which have *like directions* are said to be **parallel.** Thus parallel straight lines never meet, however far they are produced.

3. The *test* of like directions may be thus explained :



In the above figure, AB and CD are two straight lines, and a third straight line PQ is drawn to cut them at X and Y.

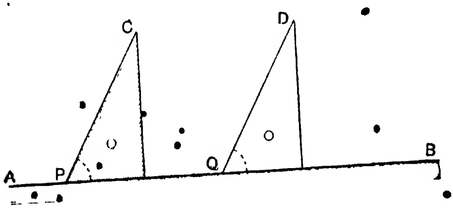
Then the direction of AB, as compared with PQ, is fixed by the angle QXB ; and the direction of CD, as compared with PQ, is fixed by the angle QYD.

Now if the angle QXB = the angle QYD, the lines AB and CD have *like directions* as compared with PQ, and will never meet, however far they are produced. That is, AB and CD are parallel.

The line PQ, which is drawn across AB and CD, is called a **transversal.**

The angles QXB, QYD, which fix the directions of AB, CD in comparison with PQ, are called **corresponding angles.**

4. We now see how we can draw parallel straight lines with a triangular ruler (known as a **set square**) and a straight ruler.



Place a set square in any position such as that shaded in the diagram, and against one of its sides lay a straight ruler (marked AB in the figure). Holding the ruler firm, slide the set square along it, so that the side marked PC moves into the position QD. Then QD and PC are parallel. For the *corresponding angles* BQD, BPC are merely different positions of the same angle of the set square.

Ex. 1. By what angle does the minute-hand of a clock change its direction in $7\frac{1}{2}$ minutes?

Ex. 2. The wind shifts from West to North-West. What angle (in degrees) measures its change of direction?

Ex. 3. From two points on the shore observers turn their glasses on a boat at sea. Are they looking in the same direction? If not, make a sketch to represent the difference of the directions. Is this difference of direction the same whether the boat is near or far away? Illustrate by a sketch.

Ex. 4. A ship steaming due East alters her course 25° towards North. Make a rough sketch to represent this, marking the angle which shows her change of direction.

(N.B. The change of direction in this case is given by the angle between the new course and *the line which would have been followed if the ship had gone straight on.*)

Ex. 5. A man walks due South, then turns 43° towards West. Make a sketch to show his first direction, his second direction, and his change of direction. [See note to Ex. 4.]

Ex. 6. Draw a straight line AB, and take two points P and Q in it. With your protractor draw PC and QD each perpendicular to AB. Are PC and QD parallel? Why?

Ex. 7. Two men walk along a straight road. At different points in the road each changes direction 37° to his left. Make a rough sketch to show each change of direction, and say why the new paths are parallel.

Ex. 8. Three ships in line ahead steam due West, then all change course 25° towards South. Show by a sketch each change of direction, and say why the new courses are parallel.

Ex. 9. In a straight line AB take any two points P and Q, and with your protractor draw two lines PC, QD, making the $\angle$BPC = 78°, and the angle BQD = 34°. Through how many degrees must QD be turned about Q to become parallel with PC?

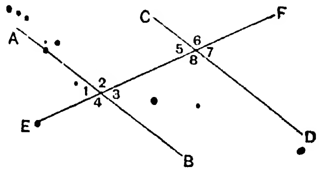
Ex. 10. Repeat the last Exercise, this time making the $\angle$BPC = 125°, and the $\angle$BQD = 51°. (i) Through how many degrees must QD turn to become parallel with PC? (ii) Through how many degrees must PC turn to be parallel with QD?

Ex. 11. I walk due North, turn 25° East, and later turn 25° towards North. Show both changes of direction by a sketch, and prove that my first and last courses are parallel.

Ex. 12. Two straight rods AP, BQ, turn the same way and at the same rate about fixed pivots A and B. If at starting the rods were in the same line with AB, show that in all subsequent positions they will be parallel. Consider the two cases (i) when the rods pointed the same way at starting; (ii) when they pointed opposite ways.

(Corresponding, Alternate, and Interior Angles.)

5. Draw with set squares two parallel lines AB, CD, and rule *any* straight line EF across them. Then number the angles so formed as in the figure below.



(i) Point out four pairs of **corresponding** angles.

The two angles in each pair (for instance, 1 and 5) may be proved equal. For the angles 1 and 5 fix the directions of AB and CD relatively to EF; and since AB and CD are parallel these directions are *alike*. That is, the angle 1 = the angle 5.

(ii) The angles 3 and 5 are said to be **alternate**. Point out another pair of alternate angles.

If AB, CD are parallel, alternate angles (for instance, 3 and 5) may be proved equal.

For the angle 3 = the angle 1; (Why ?)
and the angle 1 = the angle 5; (Why ?)
therefore the angle 3 = the angle 5.

(iii) The angles 3 and 8 are called **interior** angles. So also the angles 2 and 5 are interior.

Measure the angles 3 and 8, and add together the results. If AB, CD are parallel, you should find that

the angle 3 + the angle 8 = 180°.

This may be accounted for as follows:

The angle 3 = the angle 7; (Why ?)

∴ the angle 3 + the angle 8 = the angle 7 + the angle 8
= 2 right angles. (Why ?)

Ex. 13. Draw two parallels AB, CD with your set squares. With your protractor draw a transversal EF, making an angle of 57° with AB. Call this angle 1, and number the rest as before. Now write down (without measurement) the number of degrees in each of the angles 2, 3, 4, 5, 6, 7, 8.

Ex. 14. Repeat the last Exercise, drawing EF at an angle of 117° with AB. Write down (without measuring) the remaining angles.

Ex. 15. Again go through the last Exercise, making EF cut CD at an angle of 75°. As before, write down (without measuring) the remaining angles.

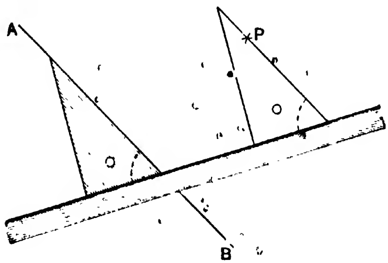
Ex 16. Draw a line AB about $3\frac{1}{2}''$ long. Take a point P about 2″ from AB. From A draw a line through P, and measure the angle PAB. Now, using your protractor, draw a line through P parallel to AB. Do this in two ways: (i) by making *corresponding* angles equal; (ii) by making *alternate* angles equal.

VI ON THE USE OF SET SQUARES.

(Parallels and Perpendiculars.)

Observe that in each set square one angle is a *right angle*. The remaining angles in one set square are both 45°; in the other they are 60° and 30°.

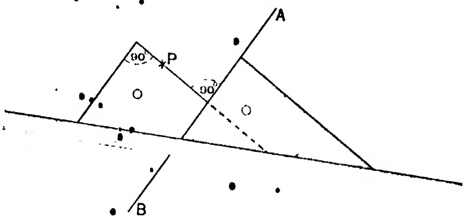
1. *Through a given point* P *to draw* **with a set square** *a line parallel to a given straight line* AB.



Place either set square so that one of its sides lies along AB in the position shaded in the diagram.

Against either of the other sides lay a straight-edge; then slide the set square along it until the side originally placed along AB passes through the point P. A line ruled along this side is parallel to AB. State the reason.

2. *Through a given point* P *to draw* **with set squares** *a line* **perpendicular** *to a given straight line* AB.



Take either set square and place one of the sides containing the right angle along AB.

Apply the straight-edge to the longest side (*i.e.* the side opposite the right angle) of the set square: and slide the latter until the side originally perpendicular to AB passes through P.

A line ruled along this side will be perpendicular to AB, for the alternate angles marked in the diagram are equal.

Ex. 1. Take two points A and B, 6 cm apart. Through A draw any straight line; and through B draw a parallel line with your set squares.

Ex. 2. Draw a line AB of length 3″. With your protractor draw AC making an angle of 76° with AB. Now through B draw a line parallel to AC.

Ex. 3. Repeat Ex. 2, making AB of length 9 cm., and the ∠BAC equal to 32°. Draw the parallel through B with your set squares.

Ex. 4. Draw a right angle AOB with your protractor, making each of the arms OA, OB, 7·5 cm. in length. Through A draw a parallel to OB, and through B draw a parallel to OA. Do this with set squares.

What is the shape of the figure you have just drawn?

Ex. 5. Draw a straight line AX, and mark off along it AB, BC, CD, each 1″ in length. Through A, B, C, and D draw lines perpendicular to AX. Why are these lines parallel?

Ex. 6. Draw a line AB of length 7 cm. Through A draw a perpendicular to AB, and along it measure AC 7 cm. long. Through B draw a parallel to AC; and through C draw a parallel to AB.

What is the shape of the figure you have thus drawn?

Ex. 7. Draw a line AB, 8 cm. long. Draw AC perpendicular to AB, and make AC = 6 cm. Join BC. From A draw AD perpendicular to BC.

Ex. 8. Draw a line AB. Through A draw a line (with your set squares) making with AB (i) a right angle, (ii) an angle of 60°, (iii) an angle of 30°, (iv) an angle of 45°.

Ex. 9. Following the principle of Art. 2, devise arrangements of a set square and straight edge by which a line may be drawn through any point P making with a given line AB an angle (i) of 45°, (ii) of 60°, (iii) of 30°.

Ex. 10. Draw a straight line AB, and take any point P outside it. Draw PX perpendicular to AB (with set squares). Measure PX.

Now take any two points Y, Z in AB on the same side of X. Join and measure PY, PZ.

Of the lines PX, PY, PZ, which is least? Which is greatest? Can you draw from P to AB a shorter line than the perpendicular PX?

3. The distance of a point P from a straight line AB is understood to be the *length of the perpendicular* PX, this being the shortest line that can be drawn from P to AB.

A point P is said to be *equidistant* from two straight lines AB, CD when the perpendiculars drawn from P to the two lines are equal.

Plane Figures. Definitions.

1. Any portion of a plane surface bounded by one or more lines is called a **plane figure.**

The sum of the bounding lines is called the **perimeter** of the figure. The amount of surface enclosed by the perimeter is called the **area.**

2. **Rectilineal figures** are those which are bounded by straight lines.

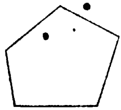
3. A **triangle** is a plane figure bounded by *three* straight lines.

Thus it has three *sides*, three *angles*, and three angular points or *vertices*.

If a triangle has *all* its sides equal it is said to be **equilateral;**
if it has *two* sides equal, it is called **isosceles**;
if no two of its sides are equal, it is called **scalene.**

In an isosceles triangle the *vertex* is usually understood to be the point at which the *equal* sides meet, then the opposite side is called the *base*.

4. A **quadrilateral** is a plane figure bounded by *four* straight lines.



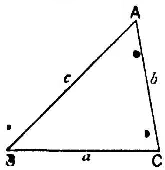
5. A **polygon** is a plane figure bounded by more than four straight lines.

6. A rectilineal figure is said to be

equilateral, when all its sides are equal;
equiangular, when all its angles are equal;
regular, when it is both equilateral and equiangular.

VII. Construction and Comparison of Triangles.

1. The three sides and three angles of a triangle are called its six **parts.** A triangle may also be considered with regard to its **area.**



In the triangle ABC the letters A, B, C are used not only to name the vertices, but to represent the size of the angles as measured in degrees; while a, b, c are taken to represent the lengths of the opposite sides.

Thus in the figure $\begin{cases} A = 58^\circ, \quad B = 44^\circ, \quad C = 78^\circ; \\ a = 2{\cdot}6 \text{ cm.}, \ b = 2{\cdot}1 \text{ cm.}, \ c = 3{\cdot}0 \text{ cm.} \end{cases}$

The symbol $\triangle$ is used as an abbreviation for the word *triangle.*

(Angles of a Triangle.)

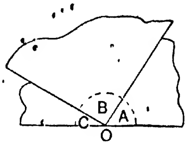
2. Let the pupil now draw three or four triangles varying in size and shape, and in each triangle let him measure the three angles and find their sum. Allowance being made for small errors in measurement, the results should suggest that in each of the cases examined

$$A + B + C = 180^\circ.$$

It will be proved later [Theorem 7, p. 54] that this is true of every triangle; namely that *the sum of the three angles is equal to* **two right angles**. For the present the truth of this will be assumed.

NOTE. The following verification by experiment is worth notice.

Draw a good-sized triangle *of any shape you like*. Cut it out and tear off the corners. Fit these together at a point O; and observe the two outer edges. It will be seen that these lie in a straight line. In other words, the three angles of the triangle together make up a *straight angle*, that is, two right angles.



3. A triangle is said to be **right-angled** when *one* of its angles is a right angle.

In a right-angled triangle the side opposite to the right angle is called the **hypotenuse**.

A triangle is said to be **obtuse-angled** when *one* of its angles is obtuse.

A triangle is **acute-angled** when *all three* of its angles are acute.

(*Most of the following questions may be answered orally.*)

Ex. 1. Draw a *right-angled* triangle; an *obtuse-angled* triangle; an *acute-angled* triangle.

Can a triangle have more than one right angle?

How many acute angles must every obtuse-angled triangle have?

Why would it not be enough to say, "A triangle is acute-angled when *one* of its angles is acute"?

Ex. 2. In a $\triangle$ ABC.

(i) If A = 70°, B = 50°, how many degrees are there in C?

(ii) If B = 28°, C = 112°, find A.

(iii) If C = 126°, A = 33°, find B.

(iv) If B = 97°, A = 56°, find C.

Ex. 3. In a triangle ABC *right-angled* at B,

(i) If A = 47°, find C.

(ii) If C = $\frac{2}{3}$ of one right angle, what fraction of a right angle is A ?

(iii) If A = C + 10°, find A and C.

Ex. 4. In a triangle ABC,

(i) If A = $\frac{1}{4}$ of a *straight* angle, B = $\frac{3}{5}$ of a *straight* angle, find the number of degrees in C.

(ii) If B = $\frac{3}{10}$ of a right angle, C = $\frac{1}{10}$ of a right angle, what fraction of one right angle is A ?

(iii) If C = $1\frac{1}{3}$ of a right angle, A = $\frac{1}{4}$ of a right angle, what fraction of a right angle is B ?

Ex. 5. If the three angles of a triangle are equal, what fraction of a right angle is each one ?

Ex. 6. If one angle of a triangle = 110°, and the other two are equal to one another, how many degrees are there in each one ?

Ex. 7. In a triangle ABC, A = 41°, and B = 3C ; find B and C.

Ex. 8. The angles of a triangle contain respectively x, $2x$, $3x$ degrees ; find each one.

Ex. 9. Is it possible that a triangle should have the following angles ?

(i) 85°, 78°, 17° ; (ii) 102°, 43°, 36° ; (iii) 47°, 83°, 49°.

Ex. 10. In a △ ABC,

(i) If A + B = C, find the angle C.

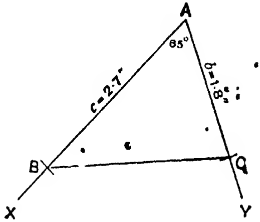
(ii) If B + C = 110°, B − C = 20°, find A, B, and C.

(Construction and Congruence of Triangles.)

4. Before constructing a triangle or other figure from given sides and angles, it is very useful to draw a rough free-hand sketch, in order to make sure that the question is understood, and to show what is given and what is required. The *data*, or *things given*, should be written on the sketch.

5. CONSTRUCTION I. *To draw a triangle having given* **two sides** *and the* **included angle.**

(For instance $b=1{\cdot}8''$, $c=2{\cdot}7''$, $A=65°$.)



Construction. Draw a line AX; and from A draw AY making an angle of 65° with AX (using protractor).

From AX cut off AB equal to 2·7" (the length of c).

From AY cut off AC equal to 1·8" (the length of b).

Join BC.

Then ABC is evidently the required triangle.

Ex. 11. Draw a triangle ABC, having given

(i) AB = 6·2 cm., AC = 7·8 cm., and the included angle BAC = 118°.

(ii) AC = 4·2", AB = 2·8", and the ∠A = 40°.

(iii) BC = 9·5 cm., BA = 11·8 cm., and the ∠B = 35°.

(iv) a = 9·1 cm., b = 14·0 cm., and the ∠C = 18°.

Ex. 12. Draw two triangles ABC, PQR, having given

AB = PQ = 2·6", AC = PR = 4·8", ∠A = ∠P = 46°.

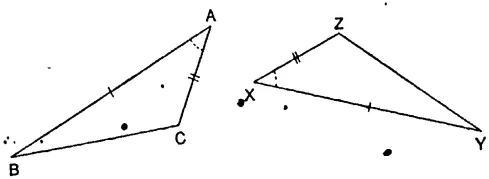
Carefully compare the resulting triangles as regards shape and size. Measure BC and QR.

Ex. 13. Draw two triangles ABC, XYZ, having given

AC = XZ = 3·7", CB = ZY = 2·5", ∠C = ∠Z = 67°.

Compare the resulting triangles. Measure the angles A and X, also the angles B and Y.

6. Observe carefully in the foregoing Exercises that if we are given the lengths of *two sides* and the size of the *included angle*, the resulting triangle is *completely fixed in size and shape*; so that all triangles drawn from these given parts (however differently placed) must be exactly alike.



We conclude that in two triangles ABC, XYZ, if AB = XY, AC = XZ, and the included ∠A = the included ∠X, then the triangles are alike in all respects; and either of them (or a copy of it made on tracing paper) may be so placed upon the other as exactly to fit over it. So that the remaining parts of the two triangles must be equal, each to each.

The process of fitting one figure over another for the purpose of comparison is called **superposition**, and the first figure is said to be **applied** to the other.

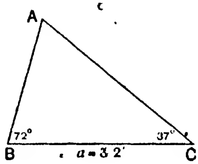
In two such triangles corresponding sides are *opposite to equal angles*; and corresponding angles are *opposite to equal sides*.

7. Triangles which, being alike in all respects, may be made to coincide by superposition are said to be **congruent**.

Thus in congruent triangles each part of the first is equal to the **corresponding** part (namely that with which it coincides) of the other; and the triangles are equal in area.

8. CONSTRUCTION II. *To draw a triangle having given* **one side** *and the* **two angles** *at its ends.*

(For instance $a = 3{\cdot}2''$, $B = 72^\circ$, $C = 37^\circ$.)



Construction. Draw BC equal to $3{\cdot}2''$.

At B make an angle of 72° with BC (using protractor).

At C make an angle of 37° with CB, on the same side as before.

Produce the lines to meet at A.

Then ABC is the required triangle.

NOTE. When *two angles* are among the data it must be remembered that the third angle is known *before* the triangle is constructed. A rough free-hand sketch should be made and the three angles written into it before drawing the triangle with rule and protractor.

Ex. 14. Draw a triangle ABC from the following data, and in each case write down the third angle:

(i) $a = 6{\cdot}8$ cm., $\angle B = 101^\circ$, $\angle C = 44^\circ$.

(ii) $\angle C = 84^\circ$, $\angle A = 35^\circ$, $b = 5{\cdot}5$ cm.

(iii) $\angle B = 36^\circ$, $c = 3{\cdot}0''$, $\angle A = 104^\circ$.

(iv) $a = 3{\cdot}5''$, $\angle A = 47^\circ$, $\angle B = 43^\circ$.

(v) $\angle B = 70^\circ$, $b = 11{\cdot}0$ cm., $\angle C = 37^\circ$.

Ex. 15. Try to draw triangles in which

(i) $a = 5{\cdot}8$ cm., $B = 110^\circ$, $C = 70^\circ$;

(ii) $a = 5{\cdot}8$ cm., $B = 45^\circ$, $C = 135^\circ$.

What difficulty arises? Perhaps you find that the other sides would not meet on your paper: would they *ever* meet? Give a reason for your answer.

Ex. 16. Construct two triangles ABC, PQR from the following data, first writing down the third angle in each triangle:

$BC = QR = 3{\cdot}6''$; $\angle B = \angle Q = 20^\circ$; $\angle C = \angle R = 128^\circ$.

Compare the triangles as regards size and shape. Measure AB and PQ; also AC and PR.

Ex. 17. Construct two triangles ABC, XYZ, having given

$AC = XZ = 4{\cdot}5''$; $\angle B = \angle Y = 78^\circ$; $\angle C = \angle Z = 69^\circ$.

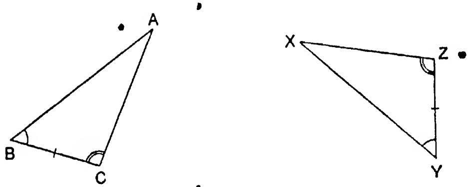
Observe that it is first necessary to find the angles A and X.

Compare the triangles, and measure AB and XY, also BC and YZ.

9. From the above Exercises it will be seen:

(i) That a triangle cannot always be drawn having two angles of given size. The sum of the two given angles must in fact be less than 180°.

(ii) That when the length of *one side* and the *two angles* at its ends are given, then (provided that such a triangle can be drawn at all) its *size and shape will be completely fixed*; and all triangles drawn from these data must be exactly alike.



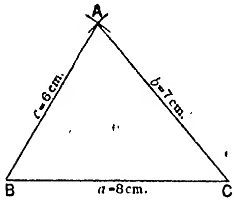
Hence we conclude that *in two triangles* ABC, XYZ, *if* BC = YZ, *and the* $\angle B$ = *the* $\angle Y$, *and the* $\angle C$ = *the* $\angle Z$, *the two triangles are* **congruent.**

NOTE. Observe that the given side in one triangle must *correspond* to the given side in the other; that is, these sides must be opposite to equal angles, in order that the triangles may be congruent.

10. We have already seen (Axiom (iii), p. 3) that *the straight line joining two points is the shortest distance between them.* This may be illustrated in *any* triangle ABC; for clearly BA + AC is greater than BC; AB + BC greater than AC; and BC + CA greater than BA. In other words, *any two sides of a triangle must be together greater than the third side.*

11. CONSTRUCTION III. *To draw a triangle, having given the three sides.*

(For instance: $a = 8$ cm., $b = 7$ cm., $c = 6$ cm.)



Construction. Draw a straight line BC of length 8 cm.

With centre B, and a radius of 6 cm, (the length of c), draw a circle.

With centre C, and a radius of 7 cm. (the length of b), draw a second circle cutting the first at A.

(*Arcs* of these circles, showing the cutting point, are enough in practice.)

Join AB and AC.

Then ABC is the triangle required.

NOTE.—Observe that the problem is the same as that of finding a point A distant 6 cm. from B, and 7 cm. from C. Can more than one such point be found?

Ex. 18. Draw *two* triangles, one on each side of BC, having the dimensions given above.

Cut out the double figure so formed, and fold it about BC. What do you find? Are the two triangles of the same size and shape?

Ex. 19. Construct triangles whose sides have the following lengths:

(i)	$a = 3{\cdot}0''$,	$b = 3{\cdot}0''$,	$c = 3{\cdot}0''$.
(ii)	$c = 3{\cdot}5''$,	$a = 2{\cdot}5''$,	$b = 3{\cdot}0''$.
(iii)	$b = 5{\cdot}4$ cm.,	$c = 7{\cdot}6$ cm.,	$a = 5{\cdot}4$ cm.
(iv)	$a = 4{\cdot}3$ cm.,	$b = 8{\cdot}2$ cm.,	$c = 5{\cdot}4$ cm.

Ex. 20. Is it possible to construct triangles having the following sides? If in any case it is impossible, carry the construction as far as it will go, and say why it fails.

(i) $a = 3{\cdot}0''$, $b = 1{\cdot}5''$, $c = 1{\cdot}0''$.

(ii) $a = 3{\cdot}0''$, $b = 2{\cdot}0''$, $c = 1{\cdot}0''$.

(iii) $c = 11{\cdot}8$ cm., $a = 7{\cdot}8$ cm., $b = 6{\cdot}0$ cm.

(iv) $a = 4{\cdot}5$ cm., $b = 7{\cdot}0$ cm., $c = 2{\cdot}5$ cm.

Ex. 21. Draw two triangles ABC, XYZ from the following data:

AB = XY = 5·0″, BC = YZ = 4·0″, CA = ZX = 3·2″.

Compare the two triangles, and measure the angles B and Y; also the angles C and Z. Deduce (without measurement) the angles at A and X.

Ex. 22. Draw two triangles ABC, YZX from the following data:

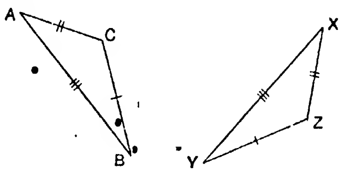
AB = YZ = 12·0 cm., BC = ZX = 8·0 cm., CA = XY = 5·0 cm.

Can a tracing of the triangle YZX be fitted exactly over the triangle ABC?

12. From the above Exercises it appears:

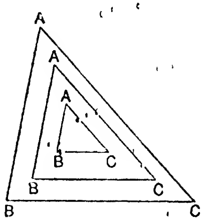
(i) That a triangle can be drawn to have sides of three given lengths only if any two of the given lengths are together greater than the third.

(ii) That, given the lengths of *the three sides*, the resulting triangle *is completely fixed in size and shape* (provided that a triangle with such sides can be drawn at all); and all triangles drawn with sides of these lengths must be exactly alike.



Hence we conclude that *two triangles* ABC, XYZ, *which have the* THREE SIDES *of one equal to the* THREE SIDES *of the other* (each to each), *must be congruent: and the corresponding angles must be equal.*

13. Lastly, would the size and shape of a triangle ABC be fixed, if we were given the *three angles*?



First we notice that the sum of the three angles (expressed in degrees) must be 180°, otherwise no triangle could be drawn from them.

Let us take A = 55°, B = 80°, C = 45°. Draw a line BC of *any* length as base Make the angle B equal to 80°, and the angle C equal to 45°: then *whatever length* we take for the base BC, the third angle A must be 55°.

Thus any number of triangles of *different sizes* can be drawn having given the three angles The figure will suggest that all these triangles have the *same shape*: in fact the three angles fix the *shape* but not the *size* of a triangle.

14. We may now sum up the conclusions so far drawn from the construction and comparison of triangles.

A triangle is completely fixed in size and shape if the following parts are given:

CONSTRUCTION I. *Two sides and the included angle.*

CONSTRUCTION II. *One side and the two angles at its ends.*

CONSTRUCTION III. *The three sides.*

Hence we conclude that *two triangles are congruent*, if we know that the three parts named in I, *or* II, *or* III are severally equal in the two triangles. But two triangles are not necessarily congruent when *any* three parts of one are respectively equal to the corresponding parts of the other.

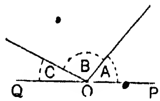
[*Formal proofs of the above statements will be given in Theorems* 9, 10, 13 (*see pp.* 60, 61, 67). *and Exercises on the application of congruent triangles will there be found.*]

(Further Exercises on the Construction and Congruence of Triangles.)

Ex. 23. Draw triangles (where possible) from the following data. When the construction fails, say why. Mention any case in which more than one triangle may be drawn.

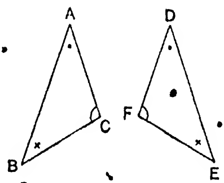
(i)	$a = 8$ cm,	$b = 6$ cm,	$c = 11$ cm.
(ii)	$b = 6$ cm,	$a = 8$ cm,	$\angle C = 105°$.
(iii)	$a = 8$ cm,	$\angle B = 53°$,	$\angle C = 127°$.
(iv)	$c = 11$ cm,	$\angle B = 30°$,	$\angle A = 45°$.
(v)	$c = 11$ cm,	$b = 6$ cm,	$a = 5$ cm.
(vi)	$\angle C = 21°$,	$\angle A = 104°$,	$\angle B = 55°$.
(vii)	$\angle A = 93°$,	$\angle C = 46°$,	$\angle B = 31°$.
(viii)	$\angle B = 45°$,	$\angle C = 100°$,	$c = 9.5$ cm.
(ix)	$\angle A = 90°$,	$\angle C = 51°$,	$a = 10.0$ cm.
(x)	$a = 8$ cm,	$b = 5$ cm,	$\angle C = 179°$.

Ex. 24. Draw a straight line PQ of any length, and take a point O in it. From O draw any two lines on the same side of PQ, and call the angles so formed A, B, and C.



With your protractor measure the angles at B and C; and on a base of 3.4″ construct a triangle having the angles at each end of the base equal to B and C. How do you know that the third angle of this triangle must be equal to A?

15. It should be noticed that in order to make two congruent triangles coincide, it may be necessary to *reverse* one of them, that is, to turn it over before superposition. This is illustrated in the adjoining figure, which shows two triangles ABC, DEF alike in size and shape, but so related that the triangle DEF must be reversed before it can be made to fit over the triangle ABC.



(Congruence.)

Ex. 25. Draw a good-sized triangle ABC of any shape; then state three different methods by which (after necessary measurements) an exact copy of it may be made.

Make a copy of the given triangle ABC in each of these ways; and test by seeing if a tracing of the triangle ABC can be exactly fitted over each copy.

Ex. 26. Draw a triangle ABC, having given $a = 8{\cdot}0$ cm., $b = 10{\cdot}0$ cm., $c = 12{\cdot}5$ cm. Measure the $\angle$A, and make a copy of the $\triangle$ABC by Construction I. Measure the $\angle$B, and make a second copy of the $\triangle$ABC by Construction II.

Ex. 27. Draw a triangle ABC, having given $\angle A = 57°$, $b = 3{\cdot}0''$, $c = 2{\cdot}5''$. Measure a, and make a copy of the $\triangle$ABC by Construction III. Measure $\angle$B, and make a second copy of the $\triangle$ABC by Construction II.

Ex. 28. From which of the conditions given below may we conclude that the $\triangle$s ABC, RST are congruent? Illustrate each case by a figure (a free-hand sketch will be sufficient) in which the parts given equal are shown by distinguishing marks:

(i) $\begin{cases} AB = RS, \\ AC = RT, \\ \angle A = \angle R \end{cases}$ (ii) $\begin{cases} AB = RT, \\ AC = RS, \\ \angle A = \angle R \end{cases}$ (iii) $\begin{cases} AB = RS, \\ \angle A = \angle R, \\ \angle B = \angle S. \end{cases}$

(iv) $\begin{cases} AC = RT, \\ \angle A = \angle R, \\ \angle C = \angle T. \end{cases}$ (v) $\begin{cases} BC = ST, \\ \angle B = \angle T, \\ \angle C = \angle S. \end{cases}$ (vi) $\begin{cases} AB = RS, \\ \angle A = \angle R, \\ \angle C = \angle T \end{cases}$

(vii) $\begin{cases} AB = ST, \\ BC = TR, \\ CA = RS. \end{cases}$ (viii) $\begin{cases} BC = RS, \\ CA = ST, \\ AB = TR. \end{cases}$ (ix) $\begin{cases} \angle B = \angle S, \\ AB = RS, \\ BC = ST. \end{cases}$

(x) $\begin{cases} \angle B = \angle S, \\ \angle A = \angle R, \\ AC = RT. \end{cases}$ (xi) $\begin{cases} \angle A = \angle R, \\ \angle B = \angle S, \\ AB = ST. \end{cases}$ (xii) $\begin{cases} BC = RT; \\ \angle A = \angle R, \\ \angle C = \angle T. \end{cases}$

* *The five simple Constructions, or Problems, which are given between pages 96 and 102, should now be worked out. No proofs should be required at this stage, but the results should be verified experimentally. These constructions are to be done with ruler and compasses only.*

VIII. Exercises on Drawing to Scale and Measurement. Heights and Distances.

1. A map or plan is a small but exact flat copy of the country or ground it represents. Therefore by measuring on a map the distance between two dots which mark certain towns, we may reckon the real distance between the towns themselves, provided we know the *scale* on which the map is drawn. For instance, if 1 inch measured on the map stands for 10 miles, then 2″ stands for 20 miles; 4·5″ for 45 miles; and so on. Such a map is said to be drawn on *the scale of 10 miles to 1 inch*.

[*In the following Exercises plans are to be drawn on squared paper ruled to tenths of an inch, and the results are to be got by measurement and reckoning.*]

Ex. 1. I walk 4 miles due North, then 3 miles due East. Draw a plan to show my journey, making 1 in. stand for 1 mile; then by measurement find how far I am from my starting point.

Ex. 2. Draw the ground-plan of a room, 30 feet long by 20 feet wide, making 1″ represent 10 feet. Find as nearly as you can the actual distance between two opposite corners.

Ex. 3. An upright pole, standing 25 feet high, is stayed by a rope carried from the top to a point on the ground 15 feet from the foot of the pole. Represent this by a drawing (scale 10 feet to 1 inch); and find the length of the rope.

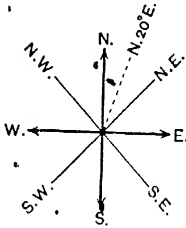
Ex. 4. A ladder reaches a window-sill 15 feet high, and the foot of the ladder rests on the ground 8 feet from the front of the house. Draw a plan (scale 5 feet to 1 inch), and use it to find the length of the ladder.

Ex. 5. Looking Eastward from my house, I see a church tower which I know to be 2 miles distant. Looking North I see a second tower $1\frac{1}{2}$ miles away. Draw a plan (scale 1 mile to 1 inch), and find how far the towers are apart.

Ex. 6. A ship on leaving harbour sails 22 miles South, then again 22 miles West. Represent her course on the scale of 10 miles to 1 inch, and find her distance from the harbour.

Ex. 7. In rowing across a river 48 metres wide, a man was carried 16 metres down stream. Represent this on a plan (scale 20 metres to 1 inch); hence find the distance between the starting-point and landing-point.

(The Points of the Compass.)



2. The line of direction which bisects the angle between North and East, is called *North-East*, and the terms North-West, South-East, South-West, have corresponding meanings.

If, looking from a light-house, a ship is seen in the direction North-West, we say that it *bears* N.W. from the light-house, or that its *bearing* is N.W. If the direction of the ship, as seen from the light-house, makes with the line pointing North an angle of 20° on the East side of that line, we say that the ship bears 20° East of North, or N. 20° E.

Ex. 8. A man walks 3 kilometres due East, then 5 kilometres due North. Draw a plan (scale 1 km. to 1 cm.), and find by measurement how far he is from his starting-point.

Ex. 9. North-West from my garden gate is a cottage, 300 yards distant: North-East of the cottage and 250 yards from it is a well. Draw a plan (scale 100 yards to 1 inch), and find as nearly as you can how far the well is from the garden gate.

Ex. 10. Two cyclists, each riding 14 km. an hour, leave a house at the same time. One goes by a straight road leading S.E.; the other by a road leading S.W. How far apart will they be in half an hour? (Scale 1 km. to 1 cm.)

Ex. 11. A man goes South 4 miles, then West 6 miles, then South again 4 miles. How far is he now from his starting-point? (Scale 2 miles to 1 inch.)

Ex. 12. A ship on leaving port sails N.W. for 18 miles, then North for 15 miles. Show her course on the scale of 10 miles to 1 inch. Find her approximate distance, and her bearing from the port, that is, how many degrees West of North.

Ex. 13. A boy walks 200 yards in a certain direction, then, turning 68° to his left, he walks 300 yards; finally he turns 68° to his right, and walks 250 yards. Show his track on a plan (100 yards to 1 inch), and explain why his third direction is parallel to his first. How far is he at last from his starting-point?

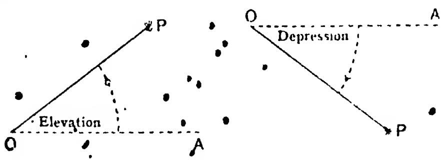
Ex. 14. A traveller wishes to go due North, but finds his way barred by a swamp. He therefore walks 5 kilometres N.E., then 5 kilometres North, then 5 kilometres N.W., and now he finds himself due North of his starting-point. How many kilometres has he lost by having gone out of his way? (Scale 1 km. to 1 cm.)

Ex. 15. A shore battery, whose guns have an effective range of 7000 yards (say 4 miles), fires on an enemy's ship bearing N.W. from the battery and distant 2½ miles. On this the ship steams N.E., 2 miles, then drops anchor, thinking herself out of range. Is she? (Scale 1 mile to 1 inch.)

3. The direction which we call **vertical** (or *upright*) is that taken by a thread from one end of which a weight hangs freely at rest. Any straight line at right angles to a vertical line is said to be **horizontal** (or *level*).

Ex. 16. How many vertical lines can pass through a given point? How many horizontal lines?

4. In the diagram given below P represents some object whose height or distance is to be found, and O the position of the observer's eye; so that OP is the *line of sight*, that is, the direction in which the object is seen. Let OA be the *horizontal* line passing from the observer's eye directly *under* or *over* the object P.



Then the $\angle AOP$ is called the **angle of elevation**, when the object is *above* the horizontal line; and the **angle of depression** when the object is below the horizontal line.

Ex. 17. A tower is observed from a point on the ground 500 feet distant from its foot, and the angle of elevation of the top is found to be 15°. What is the height of the tower? (Scale 100 feet to 1 inch.)

Ex. 18. A vertical pole, 21 feet high, is found to cast a shadow 35 feet long. How many degrees is the sun above the horizon? (Scale 10 feet to 1 inch.)

Ex. 19. A balloon, held captive by a rope 200 metres long, has drifted in the wind till its angle of elevation, as observed from the place of ascent, is 54°. How high is the balloon above the ground? (Scale 20 metres to 1 cm.)

Ex. 20. From a vessel's fore-top, 80 feet above the sea, a buoy is observed, and the angle of depression found to be 9°. How far is the buoy from the ship? (Scale 100 feet to 1 inch.)

Ex. 21. A triangular field is enclosed by two hedges and a ditch. The hedges are each 150 yards long, and they make an angle of 64°. Draw a plan (scale 50 yards to 1 inch), and find the length of the ditch.

Ex. 22. From Dover the bearing of Calais is E 31° S.; that of Boulogne is E. 63° S.; and the distances of the two French ports from Dover are respectively 23 miles and 31 miles. How far is Boulogne from Calais? (Scale 10 miles to 1 inch.)

Ex. 23. There are three towns A, B, and C. Of these, B is East of A, and distant 35 miles; while C is North of A, and distant 84 miles. A straight railway connects B and C. How far is A from the nearest point on this railway? (Scale 10 miles to 1 cm.)

Ex. 24. From a certain point on the ground I observe the top of a spire, and find the angle of elevation to be 33°. I advance 80 feet towards the spire, and then find the angle of elevation to be 47°. How high is the spire? (Scale 40 ft. to 1 inch.)

Ex. 25. A man, standing 15 feet away from the base of a monument, finds that the angle of elevation of the summit is 45°; and in making the observation his eye is 5 feet above the level of the ground. Find the height of the monument. (Scale 5 feet to 1 inch.)

THEORETICAL GEOMETRY.

Propositions.

Geometry is usually divided into a number of separate discussions, called **propositions**. Propositions are of two kinds, **Theorems** and **Problems**.

A **Theorem** proves the truth of some geometrical statement.

A **Problem** performs some geometrical construction, such as to draw some particular line, or to construct some required figure.

The preliminary statement describing the purpose of a proposition is called the **Enunciation**.

The enunciation of a theorem consists of two clauses The first clause tells us what we are to *assume*, and is called the **hypothesis**, the second tells us what *it is required to prove*, and is called the **conclusion**.

A **Corollary** is a statement the truth of which follows readily from an established proposition, and usually requires no further proof.

The letters Q.E.D. are appended to a theorem, and stand for **Quod erat Demonstrandum**, *which was to be proved*.

Note. The following symbols and abbreviations are used in the text of this book.

$\therefore$	*for* therefore,	perp	*for* perpendicular,
$=$	,, is, or are, equal to,	par^{l}	,, parallel,
$\angle$	,, angle,	par^{m}	,, parallelogram,
rt $\angle$	,, right angle,	st. line	,, straight line,
$\triangle$	,, triangle,	$\odot$	,, circle,
sq.	,, square,	$\odot^{ce}$	,, circumference;

and all obvious contractions of words commonly used, such as opp., adj., diag., etc., *for* opposite, adjacent, diagonal, etc.

[For convenience of oral work, and to prevent the rather common abuse of contractions by beginners, the above code of signs has been introduced gradually, and at first somewhat sparingly.]

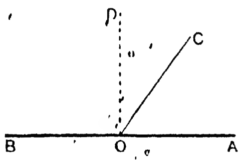
SECTION I. THEOREMS

Some of the following Theorems are marked with an asterisk. The truth of these has already been verified by observation and experiment in the Introduction, under the following heads, viz.:
IV. On Angles, V. On Parallels, VII. On the Construction and Congruence of Triangles. Whether formal proofs of all these Theorems should now be required, or whether the results obtained experimentally may be accepted as a basis for further Theorems, requiring full formal proof, must be left to the discretion of the teacher. In any case it is essential that the pupil should be well grounded in the subject matter of the Introduction, as above indicated, and be prepared to work out the Exercises attached to the marked Theorems.

ON LINES AND ANGLES.

* THEOREM 1

When a straight line meets another straight line, the adjacent angles so formed are together equal to two right angles.



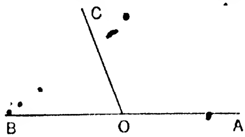
That is, if the straight line CO meets the straight line AB, forming the adjacent $\angle$s AOC, COB,

then the $\angle$AOC + the $\angle$COB = two right angles.

For, as we have already seen, *whatever be the position of* OC, the adjacent $\angle$s AOC, COB together make up the *straight angle* AOB: and the straight angle AOB is equivalent to the sum of the two right angles AOD, DOB shown in the figure.

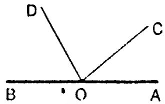
* THEOREM 2.

If two adjacent angles are together equal to two right angles, their exterior arms are in the same straight line.



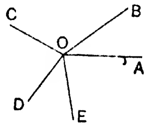
That is, if the adjacent ∠s AOC, COB are together equal to two right angles, then the exterior arms of these angles, namely OA, OB, must make a *straight angle*, so that AOB is one and the same straight line.

COROLLARY 1. *When from a point in a given straight line any number of straight lines are drawn on the same side, the sum of the consecutive angles so formed is equal to* two *right angles.*



For a straight line revolving about O, and turning in succession through the ∠s AOC, COD, DOB, will have turned through the *straight angle* AOB, that is, through *two* right angles.

COROLLARY 2. *When any number of straight lines meet at a point, the sum of the consecutive angles so formed is equal to* four *right angles.*



For a straight line revolving about O, and turning in succession through the ∠s AOB, BOC, COD, DOE, EOA, will have made *one complete revolution*, and therefore turned through four right angles.

DEFINITIONS.

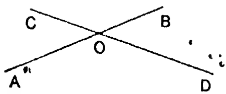
(i) Two angles whose sum is *two* right angles, are said to be **supplementary**: and each is called the **supplement** of the other.

(ii) Two angles whose sum is *one* right angle are said to be **complementary**: and each is called the **complement** of the other.

COROLLARY. (i) *Supplements of the same angle are equal.*
(ii) *Complements of the same angle are equal.*

* THEOREM 3.

If two straight lines cut one another, the vertically opposite angles are equal.



Let the straight lines AB, CD cut one another at the point O.

It is required to prove that

(i) *the* ∠ AOC = *the* ∠ DOB,

(ii) *the* ∠ COB = *the* ∠ AOD.

Proof. Because AO meets the straight line CD,

∴ the adjacent ∠s AOC, AOD together = two right angles;

that is, the ∠ AOC is the supplement of the ∠ AOD.

Again, because DO meets the straight line AB,

∴ the adjacent ∠s DOB, AOD together = two right angles;

that is, the ∠ DOB is the supplement of the ∠ AOD.

Thus each of the ∠s AOC, DOB is the supplement of the ∠ AOD,

∴ the ∠ AOC = the ∠ DOB.

Similarly, the ∠ COB = the ∠ AOD.

Q.E.D.

EXERCISES ON ANGLES.

1. Write down in degrees the supplements of 46°, 149°, 83°, 101°; also of *four-thirds* of a right angle, *five ninths* of a straight angle.

2. Write down the complements of 27°, 38°, 41°; also of *three-eighths* of a right angle.

3. Two straight lines AB, CD cut at O; if the ∠ AOC is a right angle, prove that each of the ∠s COB, BOD, DOA is a right angle.

4. Through what angles does the minute-hand of a clock turn in (i) 5 minutes, (ii) 21 minutes, (iii) $43\frac{1}{2}$ minutes, (iv) 14 min. 10 sec.? And how long will it take to turn through (v) 60°, (vi) 222°?

5. A clock is started at noon: through what angles will the hour-hand have turned by (i) 3.45, (ii) 10 minutes past 5? And what will be the time when it has turned through $172\frac{1}{2}$°?

6. The earth makes a complete revolution about its axis in 24 hours. Through what angle will it turn in 3 hrs. 20 min., and how long will it take to turn through 130°?

7. In the diagram of Theorem 3

(i) If the ∠AOC = 35°, write down (without measurement) the value of each of the ∠s COB, BOD, DOA.

(ii) If the ∠s COB, AOD together make up 250°, find each of the ∠s COA, BOD.

(iii) If the ∠s AOC, COB, BOD together make up 274°, find each of the four angles at O.

8. The angle formed by two straight lines OA, OB is 76°. Write down the number of degrees in the corresponding *reflex* angle. Illustrate by a figure.

9. From a point O in a straight line BA two lines OC, OD are drawn on the same side (as on p. 45, Cor. 1).

(i) Make a rough sketch when ∠AOC = 72°, and ∠DOB = 41°, and write down the number of degrees in the ∠COD.

(ii) Make a rough sketch when ∠AOC = 38°, and ∠COD = ∠DOB: and write down the value of each of the last angles.

10. From a point O three straight lines OA, OB, OC are drawn forming the consecutive ∠s AOB, BOC, COA. If the ∠AOB = 108°, and ∠BOC = ∠COA, make a sketch and find the ∠BOC.

11. Four straight lines OA, OB, OC, OD are drawn in order from a point O. If the consecutive ∠s AOB, BOC, COD, DOA contain respectively x, $2x$, $3x$, $4x$ degrees, find the number of degrees in each angle.

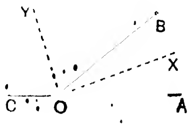
12. A straight line AOB is drawn on paper, which is then folded about O, so as to make OA fall along OB; show that the crease left in the paper is perpendicular to AB.

13. In the triangle ABC the ∠B = the ∠C. If the side BC is produced both ways, show that the exterior angles so formed are equal.

14. In the triangle ABC the ∠B = the ∠C. If AB and AC are produced beyond the base, show that the exterior angles so formed are equal.

DEFINITION. The lines which bisect an angle and the adjacent angle made by producing one of its arms are called the **internal and external bisectors** of the given angle.

Thus in the diagram, OX and OY are the internal and external bisectors of the angle AOB



EXERCISES ON ANGLES (*continued*).

15. Prove that the bisectors of the adjacent angles which one straight line makes with another contain a right angle That is to say, **the internal and external bisectors of an angle are at right angles to one another.**

16 Show that the angles AOX and COY in the above diagram are complementary

17 Show that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.

18. If the angle AOX is 35°, find the angle COY.

19. If from O, a point in AB, two straight lines OC, OD are drawn on opposite sides of AB so as to make the angle COB equal to the angle AOD, show that OC and OD are in the same straight line.

20. Two straight lines AB, CD cross at O. If OX is the bisector of the angle BOD, prove that XO produced bisects the angle AOC.

21 Two straight lines AB, CD cross at O If the angle BOD is bisected by OX, and AOC by OY, prove that OX, OY are in the same straight line.

22 Two straight lines AB, CD cross at O Prove *by the method of rotation* that the vertically opposite ∠s AOC, BOD are equal.

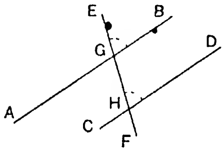
[Suppose that the line COD turns about O, and that it started from the position AOB Then since CD keeps its straightness as it turns, the same movement that opens the ∠AOC also opens the ∠BOD.; that is, these angles are the result of the same *amount of turning*, and are therefore equal to one another.]

PARALLELS.

DEFINITION. Straight lines in the same plane which have *like directions*, relatively to any given line, are said to be **parallel.**

Such lines do not meet however far they are produced, for they could only meet if they had *different directions* (page 19, Art. 1).

The pupil has already verified by experiment and observation the truth of the following statements (see pp. 20, 21).



(i) The directions of two straight lines AB, CD as compared with a third straight line EF, which cuts them at G and H, are fixed by the **corresponding** angles EGB, EHD.

(ii) If the corresponding angles are equal, AB and CD have *like directions* as compared with EF, and *will never meet if produced.*

(iii) Conversely, if AB, CD never meet if produced, they have *like directions* as compared with *any* line EF that cuts them; so that the corresponding angles EGB, EHD are equal.

These statements we shall now regard as self-evident, and express them in the following Axioms.

AXIOMS ON PARALLELS.

1. *When two straight lines are cut by another straight line, if a pair of* **corresponding angles** *are equal, the two straight lines are* **parallel.**

2. *When two* **parallel** *straight lines are cut by another straight line, the* **corresponding angles** *so formed are equal.*

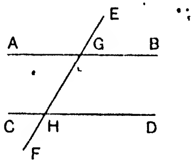
* THEOREM 4.

When two straight lines are cut by another straight line,

(i) *if the alternate angles are equal,*

or (ii) *if the interior angles on the same side are together equal to two right angles;*

then in each case the two straight lines are parallel.



(i) Let the two straight lines AB, CD be cut by the straight line EF at G and H, and let the *alternate* ∠s AGH, GHD be equal.

It is required to prove that AB *and* CD *are parallel.*

Proof. Because the ∠AGH = the vertically opposite ∠EGB,
and the ∠AGH = the ∠GHD, *by hypothesis*;
∴ the ∠EGB = the ∠GHD:
and these are *corresponding* angles;
∴ AB and CD are parallel. *Axiom* (i), p. 49.

(ii) Let the two interior ∠s BGH, GHD be together equal to two right angles.

It is required to prove that AB *and* CD *are parallel.*

Proof.

Because the sum of the ∠s BGH, GHD = two right angles,
and the sum of the adjacent ∠s BGH, BGE = two right angles;
∴ the sum of the ∠s BGH, BGE = the sum of the ∠s BGH, GHD.
From these equals take the ∠BGH;
then the ∠EGB = the ∠GHD;
and since these are *corresponding* angles, AB and CD are parallel.

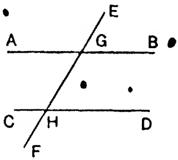
Q.E.D.

* THEOREM 5.

If a straight line cuts two parallel lines, it makes

(i) *the alternate angles equal to one another;*

(ii) *the two interior angles on the same side together equal to two right angles.*



Let the straight lines AB, CD be parallel, and let the straight line EF cut them at G and H.

It is required to prove that

(i) *the* ∠ AGH = *the alternate* ∠ GHD ;

(ii) *the two interior* ∠ˢ BGH, GHD *together* = *two right angles.*

Proof. (i) Because AB, CD are parallel, and EF cuts them,

∴ the ∠ EGB = the corresponding ∠ GHD ; *Axiom* (ii), p. 49.

but the ∠ EGB = the vertically opposite ∠ AGH ;

∴ the ∠ AGH = the alternate ∠ GHD.

(ii) Again, the ∠ EGB = the ∠ GHD ;

add to each the ∠ BGH ;

then the ∠ˢ EGB, BGH together = the ∠ˢ BGH, GHD.

But the adjacent ∠ˢ EGB, BGH together = two right angles ;

∴ the two interior ∠ˢ BGH, GHD together = two right angles.

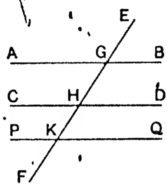
Q.E.D.

NOTES. (i) A straight line (such as EF in the above diagram) drawn across a set of given lines is called a **transversal**.

(ii) If AB is a straight line, movements from A towards B, and from B towards A are said to be in opposite **senses** of the line AB.

* THEOREM 6.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to PQ.

It is required to prove that AB *and* CD *are parallel.*

Draw a straight line EF cutting AB, CD, and PQ in the points G, H, and K.

Proof. Then because AB and PQ are parallel, and EF cuts them,

$\therefore$ the $\angle$ EGB = the corresponding $\angle$ GKQ.

And because CD and PQ are parallel, and EF cuts them,

$\therefore$ the $\angle$ GHD = the corresponding $\angle$ GKQ.

$\therefore$ the $\angle$ EGB = the $\angle$ GHD ;

and these are corresponding angles ;

$\therefore$ AB and CD are parallel.

Q.E.D.

NOTE. If PQ lies between AB and CD, the Proposition needs no proof; for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another.

AXIOM. *Through a given point there can always be one (but only one) straight line parallel to a given straight line.*

EXERCISES ON PARALLELS.

1. Two straight lines AB, CD are cut by a transversal EF at G and H (as in the figure of Theor. 4); show that AB and CD will be parallel, if

(i) the ∠EGB = the ∠FHC;

or if (ii) the ∠EGB + the ∠FHD = two right angles.

2. Two *parallels* AB, CD are cut by a transversal EF at G and H (as in the figure of Theor. 5); show that

(i) the ∠EGA = the ∠FHD;

also (ii) the ∠EGA + the ∠FHC = two right angles.

3. Two *parallels* AB, CD are cut by a transversal EF at G and H. Of the two interior angles at G and H (on the same side of EF) that at G is equal to $2\frac{1}{3}$ times that at H. Find all the angles at G and H.

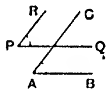
4. In a straight line AB two points P and X are taken, and straight lines PQ, XY are drawn making the ∠BPQ equal to 84°, and the ∠BXY equal to 47°. Through how many degrees must XY turn about X in order that it may become parallel to PQ? Answer this (i) when XY is supposed to turn *towards* PQ; (ii) when XY turns *away from* PQ.

5. **Straight lines which are perpendicular to the same straight line are parallel to one another.** Why?

6. **If a straight line meets two or more parallel straight lines, and is perpendicular to one of them, show that it is also perpendicular to all the others.**

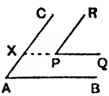
7. In the adjoining figure, PQ is given parallel to AB and the ∠QPR = the ∠BAC; show that PR is parallel to AC.

[Let PQ and AC cut at X.]



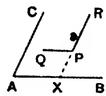
8. In the adjoining figure QP, PR are respectively parallel to BA, AC; show that the ∠QPR = the ∠BAC.

[Produce QP to meet A at X.]



9. In the marginal figure QP, PR are respectively parallel to AB, AC; show that the ∠s BAC, QPR are supplementary.

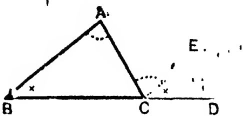
[Produce RP to meet AB at X.]



10. Two straight rods PA, QB revolve about pivots at P and Q, PA making 12 complete revolutions a minute, and QB making 10. If they start parallel and pointing the same way, how long will it be before they are again parallel, (i) pointing opposite ways, (ii) pointing the same way?

THEOREM 7.

The three angles of a triangle are together equal to two right angles.



Let ABC be a triangle.

It is required to prove that the three ∠ˢ ABC, BCA, CAB *together* = *two right angles.*

Produce BC to any point D; and suppose CE to be the line through C parallel to BA.

Proof. Because BA and CE are parallel and AC meets them,
∴ the ∠ ACE = the alternate ∠ CAB.

Again, because BA and CE are parallel, and BD meets them,
∴ the ∠ ECD = the corresponding ∠ ABC.

∴ *the whole exterior* ∠ ACD = *the sum of the two interior opposite* ∠ˢ CAB, ABC.

To each of these equals add the ∠ BCA:
then the ∠ˢ BCA, ACD together = the three ∠ˢ BCA, CAB, ABC.

But the adjacent ∠ˢ BCA, ACD together = two right angles.
∴ the ∠ˢ BCA, CAB, ABC together = two right angles.

Q.E.D.

That is to say, if A, B, *and* C *denote the number of degrees in the angles of a triangle, then*

$$A + B + C = 180°.$$

COROLLARIES TO THEOREM 7.

1. *If a side of a triangle is produced the exterior angle so formed is equal to the sum of the two interior opposite angles.*

Namely, the ext. ∠ ACD = the ∠ CAB + the ∠ ABC.

Hence the ext. ∠ ACD is *greater than* either of the int. opp. ∠s CAB, ABC.

2. *If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to the third angle of the other.*

3. *Every triangle must have at least two acute angles.*

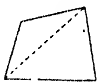
4. *In any right-angled triangle the two acute angles are together equal to one right angle.*

OTHER INFERENCES.

1. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

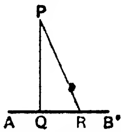
2. *The sum of the angles of any quadrilateral figure is equal to four right angles.*

[For in the adjoining figure if two opposite vertices are joined, the sum of all the angles of the two triangles thus formed = 4 right angles. And all the angles of the two triangles together make up the angles of the quadrilateral.]



3. *Only one perpendicular can be drawn to a straight line from a given point outside it.*

[If two perpendiculars could be drawn to AB from P, we should have a triangle PQR in which each of the ∠s PQR, PRQ would be a right angle, which is impossible.]



4. *Prove Theorem 7 by supposing a line to be drawn through the vertex parallel to the base.*

[For Exercises on Theorem 7, see next page. Easy questions for oral work are given on p. 28 of the Introduction.]

EXERCISES ON THEOREM 7.

1. Each angle of an equilateral triangle is two-thirds of a right angle, or 60°.

2. Two angles of a triangle are 36° and 123° respectively: deduce the third angle; and verify your result by measurement.

3. In a triangle ABC, the ∠B = 111°, the ∠C = 42°; deduce the ∠A, and verify by measurement.

4. One side BC of a triangle ABC is produced to D. If the exterior angle ACD is 134°, and the angle BAC is 42°, find each of the remaining interior angles.

5. In the figure of Theorem 7, if the ∠ACD = 118°, and the ∠B = 51°, find the ∠ A and C; and check your results by measurement.

6. ABC is a triangle in which the angles at B and C are respectively double and treble of the angle at A: find the number of degrees in each of these angles.

7. Express in degrees the angles of an isosceles triangle in which
(i) Each base angle is double of the vertical angle;
(ii) Each base angle is four times the vertical angle.

8. The base of a triangle is produced both ways, and the exterior angles are found to be 94° and 126°; deduce the vertical angle. Construct such a triangle, and check your result by measurement.

9. The sum of the angles at the base of a triangle is 162°, and their difference is 60°: find all the angles.

10. If any side of a triangle is produced both ways, the exterior angles so formed are together greater than two right angles.

11. The angles at the base of a triangle are 84° and 62°; deduce (i) the vertical angle, (ii) the angle between the bisectors of the base angles. Check your results by construction and measurement.

12. In a triangle ABC, the angles at B and C are 74° and 62°; if AB and AC are produced, deduce the angle between the bisectors of the exterior angles. Check your result graphically.

13. Three angles of a quadrilateral are respectively $114\frac{1}{2}$°, 50°, and $75\frac{1}{2}$°; find the fourth angle.

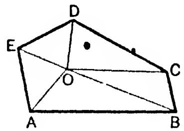
14. In a quadrilateral ABCD, the angles at B, C, and D are respectively equal to 2A, 3A, and 4A; find all the angles.

15. **If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair.**

DEFINITION. A **polygon** is a plane figure bounded by more than four sides. It is said to be **regular** when it has all its sides equal and all its angles equal.

THEOREM 8.

The sum of the interior angles of any polygon, together with four right angles, is equal to twice as many right angles as the figure has sides.



Let ABCDE be a polygon of n sides.

It is required to prove that

the sum of the interior angles + 4 rt. $\angle^s$ = $2n$ rt. $\angle^s$.

Take any point O within the figure, and join O to each of its vertices.

Then the figure is divided into n triangles.

And the three $\angle^s$ of each $\triangle$ together = 2 rt. $\angle^s$.

Hence *all* the $\angle^s$ of *all* the $\triangle^s$ together = $2n$ rt. $\angle^s$.

But all the $\angle^s$ of all the $\triangle^s$ make up all the interior angles of the figure together with the angles at O, which = 4 rt. $\angle^s$.

$\therefore$ all the int. $\angle^s$ of the figure + 4 rt. $\angle^s$ = $2n$ rt. $\angle^s$.

Hence the sum of the int. $\angle^s$ = $(2n - 4)$ *right angles.*

Q.E.D.

If D denotes the number of degrees in each angle of a *regular* polygon of n sides, the above result may be stated thus:

$$nD + 360° = n \cdot 180°.$$

EXAMPLE.

Find the number of degrees in each angle of

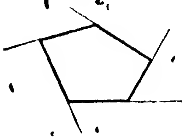
(i) a regular hexagon (6 sides);

(ii) a regular octagon (8 sides);

(iii) a regular decagon (10 sides).

COROLLARY TO THEOREM 8.

If the sides of a polygon, which has no reflex angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.



1st Proof. Suppose, as before, that the figure has n sides; and consequently n vertices.

Now at *each* vertex,

$$\text{the interior } \angle + \text{the exterior } \angle = 2 \text{ rt. } \angle^s;$$

and there are n vertices,

$\therefore$ the sum of the int. $\angle^s$ + the sum of the ext. $\angle^s = 2n$ rt. $\angle^s$.

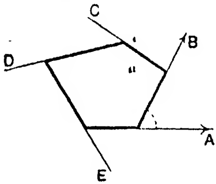
But by Theorem 8,

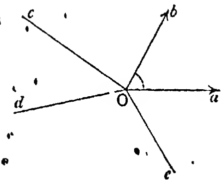
the sum of the int. $\angle^s$ + 4 rt. $\angle^s = 2n$ rt. $\angle^s$;

$\therefore$ the sum of the ext. $\angle^s = 4$ rt. $\angle^s$.

Q.E.D.

2nd Proof.





Take any point O, and suppose Oa, Ob, Oc, Od, and Oe, are lines parallel to the sides marked, A, B, C, D, E (and drawn from O in the *sense* in which those sides were produced).

Then the exterior $\angle$ between the sides A and B = the $\angle a$Ob.

And the other exterior $\angle^s$ = the $\angle^s$ bOc, cOd, dOe, eOa, respectively.

$\therefore$ the sum of the ext. $\angle^s$ = the sum of the $\angle^s$ at O

$= 4$ rt. $\angle^s$.

EXERCISES ON THEOREM 8.

1. Four angles of an irregular pentagon (5 sides) are 40°, 78°, 122°, and 135°; find the fifth angle.

2. In any *regular* polygon of n sides, each angle contains $\frac{2(n-2)}{n}$ right angles.

(i) Deduce this result from the Enunciation of Theorem 8.

(ii) Prove it independently by joining one vertex A to each of the others (except the two immediately adjacent to A), thus dividing the polygon into $n-2$ triangles.

3. How many sides have the regular polygons each of whose angles is (i) 108°, (ii) 156°?

4. Show that the only *regular* figures which may be fitted together so as to form a plane surface are (i) *equilateral triangles*, (ii) *squares*, (iii) *regular hexagons.*

5. If one side of a regular hexagon is produced, show that the exterior angle is equal to the interior angle of an equilateral triangle.

6. Express in degrees the magnitude of each *exterior* angle of (i) a regular octagon, (ii) a regular decagon.

7. How many sides has a regular polygon if each exterior angle is (i) 30°, (ii) 24°?

8. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected, show that the bisectors meet at right angles.

9. If the base of any triangle is produced both ways, show that the sum of the two exterior angles *minus* the vertical angle is equal to two right angles.

10. In the triangle ABC the base angles at B and C are bisected by BO and CO respectively. Show that the angle BOC $=90^\circ+\frac{A}{2}$.

11. In the triangle ABC, the sides AB, AC are produced, and the exterior angles are bisected by BO and CO. Show that the angle BOC $=90^\circ-\frac{A}{2}$.

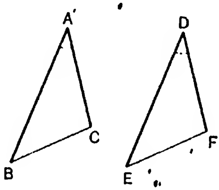
12. The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles.

TRIANGLES. CONGRUENCE.

N.B. *For a definition of Congruence, and an informal explanation of Theorems* 9, 10, and 13, *see Introduction*, pp. 30-36.

* THEOREM 9.

If two triangles have **two sides** *of one equal to* **two sides** *of the other, each to each, and the* **angles included** *by those sides equal, then the triangles are congruent.*



In the two $\triangle^s$ ABC, DEF, let AB = DE, and AC = DF, and let the included $\angle$ A = the included $\angle$ D.

It is required to prove that the $\triangle^s$ ABC, DEF *are congruent.*

Proof. Apply the $\triangle$ ABC to the $\triangle$ DEF, so that the point A falls on the point D, and the side AB along the side DE.

Then because AB = DE,

$\therefore$ the point B falls on the point E.

And because AB falls along DE, and the $\angle$ A = the $\angle$ D,

$\therefore$ AC must fall along DF

And because AC = DF,

$\therefore$ the point C falls on the point F.

Thus the points A, B, C fall on the points D, E, F;

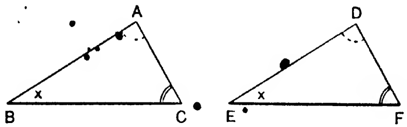
that is, the $\triangle^s$ ABC, DEF are congruent.

Q.E.D.

It follows that BC = EF, the $\angle$ B = the $\angle$ E, and the $\angle$ C = the $\angle$ F; also that the triangles are equal in area.

* THEOREM 10.

If two triangles have **two angles** *of one equal to* **two angles** *of the other, each to each, and any side of the first equal to the* **corresponding** *side of the other, the triangles are congruent.*



In the $\triangle^s$ ABC, DEF, let the $\angle$A = the $\angle$D, the $\angle$B = the $\angle$E,
also the side BC = the *corresponding* side EF.

It is required to prove that the $\triangle^s$ ABC, DEF *are congruent.*

Proof. Since the $\angle$A = the $\angle$D, and the $\angle$B = the $\angle$E,
$\therefore$ the $\angle$C = the $\angle$F. *Theor.* 7. *Cor.* 2.

Apply the $\triangle$ABC to the $\triangle$DEF, so that the point B falls on the point E, and BC along the equal side EF;
then C must fall on F.

And because the $\angle$B = the $\angle$E,
$\therefore$ BA must fall along ED.

And because the $\angle$C = the $\angle$F,
$\therefore$ CA must fall along FD.

$\therefore$ the point A, which falls both on ED and on FD, must coincide with D, the point in which these lines intersect.

Thus the points A, B, C fall on the points D, E, F;
that is, the $\triangle^s$ ABC, DEF are congruent.

Q.E.D.

It follows that AB = DE, and AC = DF; also that the triangles are equal in area.

EXERCISES ON THEOREMS 9 AND 10.

1. AB is a given straight line and O its middle point. OC is the perpendicular to AB at O, and P is any point in OC. Join PA, PB; and prove that the triangles POA, POB are congruent; hence that PA = PB.

2. ABC is an isosceles triangle. If the line which bisects the vertical angle A meets the base BC at D, prove that the triangles BAD, CAD are congruent. Hence show that

(i) BD = CD; (ii) AD is perpendicular to BC.

3. Assuming that the four sides of a square ABCD are equal, and that its angles are all right angles, prove that, if AC, BD are joined, the $\triangle$s ADC, BCD are congruent, and hence that AC = BD.

4. ABCD is a square, and L, M, and N are the middle points of AB, BC, and CD: prove that

(i) LM = MN. (ii) AM = DM.
(iii) AN = AM. (iv) BN = DM.

[Draw a separate figure in each case.]

5. ABC is a triangle right-angled at B. Produce CB to D (on the other side of AB), and make BD = BC. Join AD; and prove that the $\triangle$s ABD, ABC are congruent; and hence that

(i) AD = AC; (ii) $\angle$DAB = $\angle$CAB.

6. ABC is an isosceles triangle. From the equal sides AB, AC two equal parts AX, AY are cut off, and BY and CX are joined. Prove that the $\triangle$s XAC, YAB are congruent; and hence that

(i) BY = CX; (ii) $\angle$ABY = $\angle$ACX.

7. From the ends of a straight line AB two perpendiculars AP, BQ are drawn to AB on opposite sides; and AP, BQ are made equal. Join PQ cutting AB at O. Prove that the $\triangle$s AOP, BOQ are congruent. Hence show that O is the middle point both of AB and PQ.

8. Let O be *any* point on the bisector of the angle BAC, and from O suppose perpendiculars OP, OQ drawn to AB, AC. Prove that the $\triangle$s OPA, OQA are congruent; hence that **any point on the bisector of an angle is equidistant from the arms of the angle.**

9. Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are drawn to it from A and B: show that AX = BY; and that XY is also bisected at O.

10. PQR is an isosceles triangle of which P is the vertex. If from Q and R, the extremities of the base, QM and RN are drawn perpendicular to the equal sides PR, PQ, prove that the $\triangle$s MPQ, NPR are congruent; hence that (i) QM = RN, (ii) PM = PN; deduce MR = NQ.

Hence show that the $\triangle$s RMQ, QNR are congruent.

11. The triangle ABC is such that the line AD, which bisects the angle at A, is also perpendicular to the base BC. Prove that the triangle ABC is isosceles.

12. If in a triangle the perpendicular drawn from a vertex to the opposite side bisects that side, show that the triangle is isosceles.

13. LMN is an isosceles triangle, and P, Q are the middle points of the equal sides LM, LN. If PN, QM are joined, show that the triangles PLN, QLM are congruent; hence that MQ = NP, and ∠LPN = ∠LQM.

Deduce ∠MPN = ∠NQM; and hence show that the triangles MPN, NQM are congruent.

14. Two straight lines AB and DC cross one another at O in such a way that O is the middle point of each. If AC and BD are joined, prove that the △s AOC, BOD are congruent; and hence that AC = BD.

In the congruent triangles pick out two equal *alternate* angles, and prove that AC, BD are parallel.

15. Two men A and B part company at a place O. A travels East then North; B travels North then East. *Each* man goes as far North as the other goes East. Show this by means of a figure, and prove that they finish at equal distances from O.

16. ABCD is a square. In AB, BC, CD, DA points X, Y, Z, V are so taken that AX = BY = CZ = DV; and XY, YZ, ZV, VX are joined.

Prove that the four △s AXV, BYX, CZY, DVZ are congruent. Hence show that the figure XYZV is equilateral; and (by means of Theor. 7) that each of its angles is a right angle.

[Assume that the sides of a square are all equal, and its angles all right angles.]

17. From the ends of a straight line AB any two parallel lines AX, BY are drawn on opposite sides of AB, and AX and BY are made equal. Join XY. Pick out two equal alternate angles, and prove that the △s APX, BPY are congruent. Hence show that AB, XY bisect one another.

18. On the base AB and on opposite sides of it two △s PAB, QBA are drawn so that the ∠s at A and B in the △ PAB are respectively equal to the ∠s at B and A in the △ QBA. Call this Fig. 1.

Could these triangles be made to coincide by folding about AB? Are they congruent?

Draw the figure that would result if the △ QBA were folded about AB till it came into the plane of the △ PAB. Call this Fig. 2.

Join PQ in each figure, and prove that

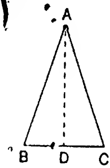
(i) the △s PAQ, QBP are congruent in both figures;

(ii) PQ bisects AB in Fig. 1: (iii) PQ is parallel to AB in Fig. 2.

19. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

THEOREM 11.

If two sides of a triangle are equal, the angles opposite those sides are equal.



Let ABC be a triangle, in which the side AB = the side AC.

It is required to prove that the ∠B = *the* ∠C.

Suppose that AD is the line which bisects the ∠BAC, and let it meet BC in D.

1st Proof. Then in the △s BAD, CAD,

because { BA = CA,
AD is common to both triangles,
and the included ∠BAD = the included ∠CAD;

∴ the triangles are congruent; *Theor.* 9.

so that the ∠B = the ∠C.

Q.E.D.

2nd Proof. Suppose the △ABC to be folded about AD.

Then since the ∠BAD = the ∠CAD,

∴ AB must fall along AC.

And since AB = AC,

∴ B must fall on C, and consequently DB on DC.

∴ the ∠B will coincide with the ∠C, and is therefore equal to it.

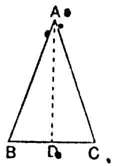
Q.E.D.

Obs. This Theorem is sometimes enunciated as follows:

The angles at the base of an isosceles triangle are equal.

Theorem 12.

If two angles of a triangle are equal, then the sides which are opposite to those angles are equal.



Let ABC be a triangle in which the $\angle$B = the $\angle$C.

It is required to prove that the side AC = *the side* AB.

Suppose that AD is the line which bisects the $\angle$BAC, and let it meet BC in D.

Proof. Then in the $\triangle$s BAD, CAD,

because $\begin{cases} \text{the } \angle B = \text{the } \angle C, & \textit{Given.} \\ \text{and the } \angle BAD = \text{the } \angle CAD, \\ \text{and AD is common to both triangles,} \end{cases}$

$\therefore$ the triangles are congruent, *Theor.* 10.

so that AC = AB. Q.E.D.

Note. If we interchange the hypothesis and conclusion of a theorem, we enunciate a new theorem which is called the **converse** of the first.

Thus in Theorem 11

we *assume* that AB = AC, and *prove* that the $\angle$ABC = the $\angle$ACB.

And in Theorem 12

we *assume* that the $\angle$ABC = the $\angle$ACB, and *prove* that AB = AC.

Hence Theorem 12 is the converse of Theorem 11, for *the hypothesis of each is the conclusion of the other.*

Definition. A figure is said to be **symmetrical about a line** when, on being folded about that line, the parts of the figure on each side of it can be brought into coincidence.

The straight line is called an **axis of symmetry.**

That this may be possible, it is clear that the two parts of the figure must have the same size and shape, and must be similarly placed with regard to the axis.

Theorem 11 proves that *an isosceles triangle is symmetrical about the bisector of its* Vertical *angle.*

EXERCISES ON THEOREMS 11 AND 12.

1. Show that if the sides of a triangle are all equal, its angles are all equal.

2. If the angles of a triangle are equal, its sides are equal.

3. ABCD is a four-sided figure whose sides are all equal, and the diagonal BD is drawn: show that

(i) $\angle ABD = \angle ADB$; (ii) $\angle CBD = \angle CDB$; (iii) $\angle ABC = \angle ADC$.

4. If *the base* of an isosceles triangle is produced both ways, show that the exterior angles so formed are equal, and obtuse.

5. If the *equal sides* of an isosceles triangle are produced beyond the base, show that the exterior angles so formed are equal, and obtuse.

6. ABC, DBC are two isosceles triangles drawn on opposite sides of the same base BC: prove (by means of Theorem 11) that $\angle ABD = \angle ACD$.

7. ABC, DBC are two isosceles triangles drawn on the same side of the same base BC: employ Theorem 11 to prove that $\angle ABD = \angle ACD$.

8. AB, AC are the equal sides of an isosceles triangle ABC; and L, M, N are the middle points of AB, BC, and CA respectively: then

(i) LM = NM; (ii) BN = CL; (iii) $\angle ALM = \angle ANM$.

9. Prove that any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

10. If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, show that the triangle thus formed is isosceles.

11. From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z: show the triangle AYZ is isosceles.

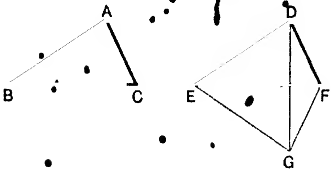
12. If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, show that the triangle is isosceles.

13. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA; if DC is drawn, show that BCD is a right angle.

14. AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected: if through any point X in DC a straight line YXZ is drawn parallel to AB and meeting the bisectors in Y and Z, show that XY is equal to XZ.

* THEOREM 13.

If two triangles have the **three sides** *of one equal to the* **three sides** *of the other, each to each, the triangles are congruent.*



In the triangles ABC, DEF, let AB = DE, AC = DF, and BC = EF.

It is required to prove that the triangles are congruent.

Proof. Apply the △ABC to the △DEF, so that B falls on E, and BC along EF, and so that A is on the side of EF opposite to D. [See Note below.]

Then because BC = EF, C must fall on F.

Let GEF be the new position of the △ABC.

Join DG.

Because ED = EG, ∴ the ∠EDG = the ∠EGD. *Theor.* 11.

Again, because FD = FG, ∴ the ∠FDG = the ∠FGD.

Hence the whole ∠EDF = the whole ∠EGF,

that is, the ∠EDF = the ∠BAC.

Then in the △s BAC, EDF;

because $\left\{\begin{array}{l}\text{BA = ED,}\\ \text{AC = DF,}\\ \text{and the included ∠BAC = the included ∠EDF;}\end{array}\right.$

∴ the triangles are congruent. Q.E.D.

Hence the *corresponding* angles are equal; namely, ∠C = ∠F, ∠B = ∠E, and ∠A = ∠D; also the triangles are equal in area.

NOTE. The *greatest* side of the △ABC is chosen for superposition so that DG falls within the ∠s EDF, EGF.

*** *At this stage Problems* 1 *to* 5, *pp.* 96-102, *should be worked out in full, the proofs, there given only in outline, affording useful exercises in Congruent Triangles.*

EXERCISES ON CONGRUENCE OF TRIANGLES.

1. Two $\triangle^s$ APB, AQB lie on opposite sides of the common base AB, and are such that AP = AQ, and BP = BQ. Prove that the triangles are congruent. Hence show that

(i) $\angle$P = $\angle$Q ; (ii) AB bisects each of $\angle^s$ PAQ, PBQ.

2. ABC is an isosceles triangle whose vertex is A, and O is the middle point of the base BC. Join AO ; and prove that the $\triangle^s$ AOB, AOC are congruent. Hence show that

(i) AO bisects the vertical angle BAC ;

(ii) AO is perpendicular to BC.

3. If ABCD is a rhombus, that is, an equilateral four-sided figure ; and if the diagonal AC is drawn, prove that the $\triangle^s$ ABC, ADC are congruent. [Theor. 13.] Hence show that

(i) AC bisects each of the $\angle^s$ BAD, BCD ;

(ii) the four $\angle^s$ BAC, DAC, BCA, DCA are equal. [Theor. 11.]

Pick out a pair of equal *alternate* angles, and prove that AB and DC are parallel.

4. If in a quadrilateral ABCD the opposite sides are equal, namely AB = CD and AD = CB, prove that the $\angle$ADC = the $\angle$ABC.

5. If ABC and DBC are two isosceles triangles drawn on the same base BC, prove (by means of Theorem 13) that the $\angle$ABD = the $\angle$ACD, taking (i) the case where the triangles are on the *same* side of BC, (ii) the case where they are on *opposite* sides of BC.

6. If ABC, DBC are two isosceles triangles drawn on opposite sides of the same base BC, and if AD be joined, prove that each of the angles BAC, BDC will be bisected.

7. If two given points in the base of an isosceles triangle are equidistant from the extremities of the base: show that they are also equidistant from the vertex.

8. Show that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

9. ABC is an isosceles triangle having AB equal to AC ; and the angles at B and C are bisected by BO and CO : show that

(i) BO = CO ; (ii) AO bisects the $\angle$BAC.

10. Show that the diagonals of a rhombus (that is, an equilateral four-sided figure) bisect one another at right angles.

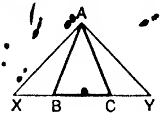
11. The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE = AF ; and FB, EC are joined : show that FB = EC.

12. ABC is an isosceles triangle, and in the base BC produced X and Y are so taken that BX = CY. Join AX, AY and prove that

(i) the △s ABX, ACY are congruent;

(ii) the △ AXY is isosceles.

Now prove in *three* different ways (namely, by Theorems 9, 10, 13) that the △s ACX, ABY are congruent.



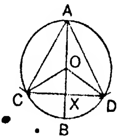
13. AB is the diameter of a circle whose centre is O. From centre A with any radius the circumference is cut (as in the figure) at C and D. Join OC, OD, AC, AD, and prove that

(i) the △s AOC, AOD are congruent;

(ii) the ∠BOC = the ∠BOD;

(iii) if CD is joined cutting AB at X, then the △s OXC, OXD are congruent;

(iv) the ∠BOC = twice the ∠BAC. [Theor. 7.]

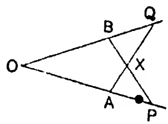


14. In the adjoining figure it is given that OP = OQ, and OA = OB. Join AQ, PB, cutting at X; and prove that

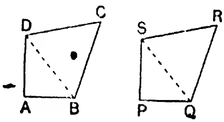
(i) the △s AOQ, BOP are congruent;

(ii) the △s AXP, BXQ are congruent.

Join AB; and prove *in three ways* that the △s ABQ, BAP are congruent.



15. In the two quadrilaterals ABCD, PQRS it is given that the four sides of one are equal to the four sides of the other, taken in order, namely AB = PQ, BC = QR, etc., and that the diagonal BD = the diagonal QS. Prove that



(i) the △s DAB, SPQ are congruent;

(ii) the △s BCD, QRS are congruent. [Theor. 13.]

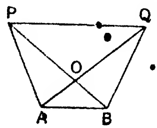
16. In the adjoining figure it is given that AP = BQ, and AQ = BP; and PQ is joined. Prove that

(i) the △s PAB, QBA are congruent;

(ii) the △s PAQ, QBP are congruent.

If AQ and BP intersect at O, show that the △s AOP, BOQ are congruent; and hence that the △s OAB, OPQ are isosceles.

Prove also that PQ is parallel to AB.



EXERCISES ON CONGRUENCE OF TRIANGLES (*continued*).

17. O is the middle point of a straight line AB which cuts two parallels LM, XY at A and B; and from O perpendiculars OP, OQ are drawn to LM and XY. Show that the $\triangle$s AOP, BOQ are congruent; hence that O is equidistant from the parallels.

If AQ and BP are joined, show that the $\triangle$s AOQ, BOP are congruent.

18. A straight line drawn between two parallels and terminated by them, is bisected; show that any other straight line passing through the middle point and terminated by the parallels, is also bisected at that point.

19. If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal.

20. In a quadrilateral, ABCD, if AB = AD, and BC = DC: show that the diagonal AC bisects each of the angles which it joins; and that AC is perpendicular to BD.

21. If the bisector of the vertical angle A of a triangle ABC also bisects the base, the triangle is isosceles.

[Let AE, the bisector of $\angle$ A, bisect the base of E. Produce AE to D, making ED equal to AE. Join DC.]

22. On the sides AB, BC of any $\triangle$ ABC equilateral $\triangle$s PAB, QBC are drawn outside the $\triangle$ ABC. Show that the $\angle$ PBC = the $\angle$ ABQ. Hence prove that the $\triangle$s PBC, ABQ are congruent.

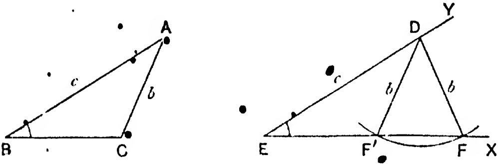
23. In a $\triangle$ ABC, in which the $\angle$ B is acute, BP is drawn perpendicular and equal to AB; and BQ is drawn perpendicular and equal to BC, both perpendiculars BP, BQ being drawn outwards from the triangle. Join PC and AQ, and prove that PC = AQ.

24. I wish to ascertain the distance between two objects A and B; but an obstacle intervenes preventing direct measurement. Having a chain for measuring lengths (but no instrument for measuring angles), I fix a mark at some point O from which both A and B are visible and accessible; and I measure OA, OB. How must I now proceed, by means of congruent triangles, to find the distance between A and B?

25. A surveyor wishes to ascertain the breadth of a river which he cannot cross. Standing at a point A near the bank, he notes an object B immediately opposite on the other bank. He lays down a line AC of any length at right angles to AB, fixing a mark at O the middle point of AC. From C he walks along a line perpendicular to AC until he reaches a point D from which O and B are seen in the same direction. He now measures CD: prove that the result gives him the width of the river.

What is known as the **Ambiguous Case** in the congruence of triangles sometimes arises when **two sides** *and* **one angle** *of one are equal respectively to* **two sides** *and* **one angle** *of the other, the given angles being* OPPOSITE *to equal sides.*

This may be explained as follows:



Let ABC be a given triangle. Let us draw another △ DEF having DE equal to AB (or c), DF equal to AC (or b), and ∠E equal to ∠B. It will appear that in this construction ambiguity arises if b is less than c.

For take a line EX of indefinite length, and at E (with protractor) make the ∠XEY equal to the ∠B

From EY cut off ED equal to BA (or c).

With centre D and radius b draw an arc cutting EX at F and F′.

Then if b is less than c, the points F, F′ will be on the same side of E; so that, by joining D to each of them, we have the *two* △ˢ DEF and DEF′ both satisfying the required conditions, but not alike in size and shape.

Hence we conclude that if in two △ˢ ABC, DEF the two sides DE, DF = the two sides AB, AC, also the ∠E = the ∠B, the triangles may, or may not, be congruent; for the shorter of the sides in the second triangle may lie in either of the positions DF or DF′.

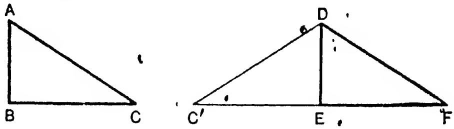
NOTE 1. It is useful to remember that *ambiguity never arises unless the shorter of the given sides is opposite the given angle.*

NOTE 2. From these data it may be shown that the angles opposite to the equal sides AB, DE are either *equal* (as for instance the ∠ˢ ACB, DF′E) or *supplementary* (as the ∠ˢ ACB, DFE); and that in the former case the triangles are congruent.

If the given angles at B and E are right angles, the ambiguity disappears. This exception is proved in the following Theorem.

THEOREM 14.

If two triangles have two sides of one equal to two sides of the other, each to each, and if the angles opposite to one pair of equal sides are **right angles**, *the triangles are congruent.*



Let ABC, DEF be two triangles in which AB = DE, and AC = DF, and the $\angle$s ABC, DEF (opposite to the equal sides AC, DF) **right angles**.

It is required to prove that the $\triangle$s ABC, DEF *are congruent.*

Proof. Apply the $\triangle$ABC to the $\triangle$DEF, so that AB falls on the equal line DE, and C on the side of DE opposite to F.

Let C′ be the point on which C falls.

Then DEC′ represents the $\triangle$ABC in its new position.

Since the sum of the $\angle$s DEF, DEC′ is two right angles,

$\therefore$ EF and EC′ are in one straight line.

And in the $\triangle$C′DF, because DF = DC′ (*i.e.* AC),

$\therefore$ the $\angle$F = the $\angle$C′. *Theor.* 11.

Hence in the $\triangle$s DEF, DEC′,

because { the $\angle$DEF = the $\angle$DEC′, being right angles;
the $\angle$F = the $\angle$C′, *Proved.*
and the side DE is common;

$\therefore$ the $\triangle$s DEF, DEC′ are congruent; *Theor.* 10.

that is, the $\triangle$s DEF, ABC are congruent.

Q.E.D.

REVISION LESSON ON CONGRUENT TRIANGLES.

1. In which of the following cases are the △s ABC, XYZ necessarily congruent ?

Illustrate each case by a free-hand sketch, and if there is congruence state the Theorem under which it falls.

(i) AB = XY, ∠A = ∠X, ∠B = ∠Y.
(ii) AB = XY, ∠A = ∠Y, ∠B = ∠X.
(iii) AB = XY, ∠A = ∠X, ∠C = ∠Z.
(iv) AB = XZ, ∠A = ∠X, ∠C = ∠Z.

2. Under what Theorems are the △s ABC, PQR congruent in the following cases ? Illustrate each case by a figure :

(i) AB = PQ = 9·0 cm., BC = QR = 6·0 cm., ∠B = ∠Q = 90°.
(ii) AC = PR = 9·0 cm., BC = QR = 6·0 cm., ∠C = ∠R = 90°.

3. ABC and PQR are two triangles in which AB = PQ = 7·0 cm.; AC = PR = 5·5 cm., and the ∠B = the ∠Q = 48°.

Show by a figure that these triangles may, or may not, be congruent; and that in the case where there is not congruence, the ∠s at C and R are supplementary.

4. In the △s ABC, PQR, AB = PQ = 2·1″, AC = PR = 2·5″; and the ∠B = the ∠Q = 85°. Prove by a figure that the triangles are congruent; and explain why *ambiguity* does not arise.

5. AB is a straight line of unlimited length, and P a point outside it. With centre P, and any sufficient radius, an arc is drawn cutting AB at X and Y. If PO is the perpendicular from P on AB, prove that O is the middle point of XY. [Theor. 14.]

6. If in two triangles the sides are equal, each to each, then the corresponding angles are also equal. Is this true ? Enunciate the Converse. Is the Converse true ?

7. From which of the conditions given below may we conclude that the triangles ABC, A′B′C′ are congruent ? Point out where ambiguity arises; and illustrate by a figure in each case.

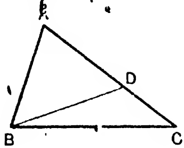
(i) A = A′ = 71°, B = B′ = 46°, $a = a'$ = 3·7 cm.
(ii) $a = a'$ = 4·2 cm., $b = b'$ = 2·4 cm., C = C′ = 81°.
(iii) A = A′ = 36°, B = B′ = 121°, C = C′ = 23°.
(iv) $a = a'$ = 3·0 cm., $b = b'$ = 5·2 cm., $c = c'$ = 4·5 cm.
(v) B = B′ = 53°, $b = b'$ = 4·3 cm., $c = c'$ = 5·0 cm.
(vi) C = C′ = 90°, $c = c'$ = 5 cm., $a = a'$ = 3 cm.

8. *Summarize the results of the above questions by stating generally under what conditions two triangles*

(i) *are necessarily congruent ;* (ii) *may or may not be congruent.*

THEOREM 15.

If one side of a triangle is greater than another, then the greater side has the greater angle opposite to it.



Let ABC be a triangle, in which the side AC is greater than the side AB.

It is required to prove that the ∠ABC *is greater than the* ∠ACB.

From AC cut off AD equal to AB.
Join BD.

Proof. Because AB = AD,
∴ the ∠ABD = the ∠ADB.

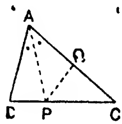
But in the △BDC, the exterior ∠ADB = the sum of the interior opposite ∠s DCB, DBC;
∴ the ∠ADB is greater than the ∠C.
∴ the ∠ABD is greater than the ∠C;
More then is the whole ∠B greater than the ∠C.

Q.E.D.

Alternative Proof. Let AP bisect the ∠A and meet BC at P.

From AC cut off AQ equal to AB, and join PQ. Then the △s BAP, QAP are congruent by Theor. 9: so that the ∠B = the ∠AQP.

But the ext. ∠AQP is greater than the ∠C;
∴ the ∠B is greater than the ∠C.



THEOREM 16.

If one angle of a triangle is greater than another, then the greater angle has the greater side opposite to it.

Let ABC be a triangle, in which the ∠ABC is greater than the ∠ACB.

It is required to prove that the side AC *is greater than the side* AB.

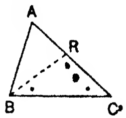
Proof. If AC is not greater than AB,
it must be either equal to, or less than AB.

Now if AC were equal to AB,
then the ∠ABC would be equal to the ∠ACB;
but, by hypothesis, it is not.

Again, if AC were less than AB
then the ∠ABC would be less than the ∠ACB; *Theor.* 15.
but, by hypothesis, it is not.

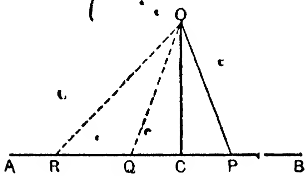
That is, AC is neither equal to, nor less than AB.
∴ AC is greater than AB. Q.E.D.

Alternative Proof. Let BR be drawn making the ∠CBR = the ∠C.
Then BR must fall within the ∠CBA, meeting AC at R;
and BR = RC by Theor. 12.
Now in the △BRA,
BR + RA is greater than AB;
∴ RC + RA is greater than AB;
that is, AC is greater than AB



Theorem 17.

Of all straight lines which can be drawn to a given straight line from a given point outside it, the perpendicular is the least.



Let OC be the perpendicular, and OP *any* oblique, drawn to the given straight line AB from the point O outside it.

It is required to prove that OC *is less than* OP.

Proof. In the $\triangle$OCP, since the $\angle$ OCP is a right angle,

$\therefore$ the $\angle$OPC is less than a right angle; *Theor.* 7, *Cor.* 3.

that is, the $\angle$OPC is less than the $\angle$OCP.

$\therefore$ OC is less than OP. *Theor.* 10.

Q.E.D.

Note. Hence conversely, if OC is the shortest line from O to AB, then OC is perpendicular to AB.

Exercises on Inequalities in a Triangle.

1. **The hypotenuse is the greatest side of a right-angled triangle.**

2. **The greatest side of any triangle makes acute angles with each of the other sides.**

3. In the above figure:

(i) If two obliques OP, OQ drawn to AB from O, make equal angles with the perpendicular OC, they are equal.

(ii) Of two obliques OQ, OR, the less is that which makes the smaller angle with the perpendicular.

4. In the triangle ABC, if $a = 3·6$ cm., $b = 2·8$ cm., $c = 3·6$ cm., arrange the angles in order of their sizes (before measurement); and prove that the triangle is acute-angled.

5. In the triangle ABC, if

(i) A = 48° and B = 51°, find the third angle, and name the greatest side.

(ii) A = B = 62½°, find the third angle, and arrange the sides in order of their lengths.

6. **If from the ends of a side of a triangle, two straight lines are drawn to a point within the triangle, then these straight lines are together less than the other two sides of the triangle.**

7. BC, the base of an isosceles triangle ABC, is produced to any point D; show that AD is greater than either of the equal sides.

8. If in a quadrilateral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle.

9. In the triangle ABC, if AC is not greater than AB, show that any straight line drawn through the vertex A and terminated by the base BC, is less than AB.

10. ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively: show that, if AB is greater than AC, then OB is greater than OC.

11. The difference of any two sides of a triangle is less than the third side.

12. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

13. The perimeter of a quadrilateral is greater than the sum of its diagonals.

14. ABC is a triangle, and the bisector of the angle BAC meets BC in X; show that BA is greater than BX, and CA greater than CX.

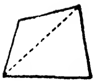
15. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

16. The sum of the diagonals of a quadrilateral is less than the sum of the four straight lines drawn from the angular points to any given point. Prove this, and point out the exceptional case.

PARALLELOGRAMS.

DEFINITIONS

1. A **quadrilateral** is a plane figure bounded by *four* straight lines.



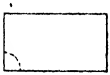
The straight line which joins opposite angular points in a quadrilateral is called a **diagonal**.

It has been shown (Theorem 7, p. 55) that the sum of the four angles of a quadrilateral = 4 right angles.

2. A **parallelogram** is a quadrilateral whose opposite sides are *parallel*.

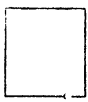
[It will be proved hereafter that the opposite sides of a parallelogram are equal, and that its opposite angles are equal.]

3. A **rectangle** is a parallelogram which has one of its angles a right angle.



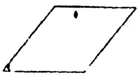
[It will be proved hereafter that *all* the angles of a rectangle are right angles. See page 83.]

4. A **square** is a rectangle which has two adjacent sides equal.

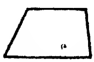


[It will be proved that *all* the sides of a square are equal and all its angles right angles. See page 83.]

5. A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

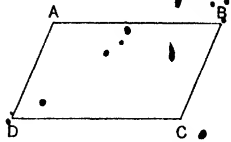


6. A **trapezium** is a quadrilateral which has *one* pair of parallel sides.



Theorem 18.

The opposite angles of a parallelogram are equal.



Let ABCD be a parallelogram in which the $\angle$s A and C are opposite, also the $\angle$s B and D.

To prove that the $\angle$ A = *the* $\angle$ C, *and the* $\angle$ B = *the* $\angle$ D.

Proof. Because AD, BC are parallel, and AB meets them,

$\therefore$ the sum of the interior $\angle$s A and B = 2 right angles.

And because AB, DC are parallel, and BC meets them,

$\therefore$ the sum of the interior $\angle$s B and C = 2 right angles.

$\therefore$ the $\angle$ A + the $\angle$ B = the $\angle$ B + the $\angle$ C,

$\therefore$ the $\angle$ A = the $\angle$ C.

Similarly, the $\angle$ B = the $\angle$ D. Q.E.D.

Conversely. *If the opposite angles of a quadrilateral are equal, it is a parallelogram.*

In the quadrilateral ABCD let the $\angle$ A = the $\angle$ C, and the $\angle$ B = the $\angle$ D.

To prove that ABCD *is a parallelogram.*

Proof. Because the $\angle$ A = the $\angle$ C, and the $\angle$ D = the $\angle$ B,

$\therefore$ the $\angle$ A + the $\angle$ D = the $\angle$ C + the $\angle$ B.

But the sum of all the interior angles of any quadrilateral = 4 right angles; *Theor.* 7.

$\therefore$ the $\angle$ A + the $\angle$ D = one-half of 4 right angles
= 2 right angles.

Now the $\angle$s A and D are the interior angles formed by AD, meeting the lines AB, DC;

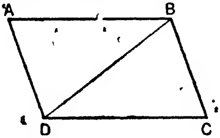
$\therefore$ AB and DC are parallel. *Theor.* 4.

Similarly, AD and BC are parallel.

$\therefore$ the figure ABCD is a parallelogram. Q.E.D.

THEOREM 19.

The opposite sides of a parallelogram are equal, and each diagonal bisects the parallelogram.



Let ABCD be a parallelogram, of which BD is a diagonal.

To prove that (i) AB = CD, *and* AD = CB;

(ii) *the* $\triangle$ ABD = *the* $\triangle$ CDB *in area.*

Proof. Because AB and DC are parallel, and BD meets them,

$\therefore$ the $\angle$ ABD = the alternate $\angle$ CDB.

Again, because AD and BC are parallel, and BD meets them,

$\therefore$ the $\angle$ ADB = the alternate $\angle$ CBD.

Hence in the $\triangle^s$ ABD, CDB,

because $\begin{cases} \text{the } \angle \text{ ABD} = \text{the } \angle \text{ CDB,} \\ \text{the } \angle \text{ ADB} = \text{the } \angle \text{ CBD,} \\ \text{and BD is common to both;} \end{cases}$ *Proved.*

$\therefore$ the triangles are congruent; *Theor.* 10.

so that AB = CD, and AD = CB; .... (i)

and the $\triangle$ ABD = the $\triangle$ CDB in area. .... (ii)

Q.E.D.

Conversely. *If the opposite sides of a quadrilateral are equal, it is a parallelogram.*

In the quadrilateral ABCD let AB = CD, and AD = CB.

To prove that ABCD *is a parallelogram.*

Proof. In the $\triangle^s$ ABD, CDB,

because $\begin{cases} \text{AB} = \text{CD, and AD} = \text{CB,} \\ \text{and BD is common to both;} \end{cases}$

$\therefore$ the triangles are congruent; *Theor.* 13.

so that the $\angle$ ABD = the $\angle$ CDB.

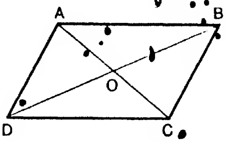
And since these are alternate angles, AB and CD are parallel.

Similarly, AD, BC may be proved parallel,

$\therefore$ ABCD is a parallelogram. Q.E.D.

THEOREM 20

The diagonals of a parallelogram bisect one another.



Let ABCD be a parallelogram of which the diagonals AC, BD intersect at O.

To prove that AO = OC, *and* BO = OD.

Proof. Because AB and DC are parallel, and AC meets them,

∴ the ∠CAB = the alternate ∠ACD.

Hence in the △s AOB, COD,

because $\begin{cases} \text{the } \angle OAB = \text{the alternate } \angle OCD, \\ \text{the } \angle AOB = \text{the vertically opposite } \angle COD, \\ \text{and } AB = \text{the opposite side } CD; \end{cases}$

∴ the triangles are congruent;

so that AO = OC, and BO = OD.

Conversely. *If the diagonals of a quadrilateral bisect one another, it is a parallelogram.*

In the quadrilateral ABCD let the diagonals AC, BD bisect one another at O.

To prove that ABCD *is a parallelogram.*

Proof. In the △s AOB, COD,

because $\begin{cases} OA = OC, \text{ and } OB = OD, \text{ by hypothesis}, \\ \text{and the } \angle AOB = \text{the vertically opposite } \angle COD; \end{cases}$

∴ the triangles are congruent;

so that the ∠OAB = the ∠OCD

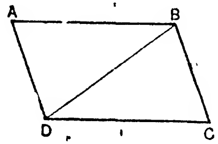
And since these are alternate angles, AB and CD are parallel.

Similarly, it may be shown that AD and BC are parallel.

∴ ABCD is a parallelogram. Q.E.D.

Theorem 21.

If a pair of opposite sides of a quadrilateral are equal and parallel, it is a parallelogram.



Let ABCD be a quadrilateral in which AB is equal and parallel to the opposite side CD.

To prove that ABCD *is a parallelogram.*

Join BD.

Proof. Because AB, DC are parallel, and BD meets them,

∴ the ∠ABD = the alternate ∠CDB.

Now in the △s ABD, CDB,

because { AB = CD, *Given.*
and BD is common to both,
and the ∠ABD = the ∠CDB; *Proved.*

∴ the triangles are congruent;

so that the ∠ADB = the ∠CBD.

But these are alternate angles;

∴ AD and BC are parallel.

And AB and DC are parallel by hypothesis;

∴ ABCD is a parallelogram.

Q.E.D.

The following inferences from Theorems 18, 19 should be noted.

1. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

That is, *all the angles of a rectangle are right angles.*

For the sum of two consecutive ∠ˢ = 2 rt. ∠ˢ; (*Theor.* 5.)
∴, if one of these is a rt. angle, the other must be a rt. angle.
And the opposite angles of the parᵐ are equal;
∴ all the angles are rt. angles.

2. *All the sides of a square are equal; and all its angles are right angles.*

EXERCISES ON PARALLELOGRAMS.

1. (i) The diagonals of a rhombus bisect one another at right angles.
(ii) Conversely, if the diagonals of a quadrilateral bisect one another at right angles, it is a rhombus.

2. (i) The diagonals of a rectangle are equal.
(ii) Conversely, if the diagonals of a parallelogram are equal, it is a rectangle.

3. If a pair of opposite angles of a parallelogram are bisected by a diagonal, it is a rhombus.

4. If the diagonals of a parallelogram are equal and bisect one another at right angles, it is a square.

5. Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point.

6. In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal.

7. A straight line PQ meets two parallels AB, CD at P and Q, and PQ is bisected at X. Through X *any* line MN is drawn to meet the parallels at M and N. Show that PM and QN are equal.

8. If ABCD is a parallelogram, and X, Y respectively the middle points of the sides AD, BC; show that the figure AYCX is a parallelogram.

9. Show that in any parallelogram ABCD the bisectors of the *opposite* angles B and D are parallel, if AB, BC are unequal.
How would the bisectors of the ∠ˢ B and D be related if AB, BC were equal?
At what angle would the bisectors of the *consecutive* angles A and B meet? Give your reason, and show that it would be the same angle for all parallelograms.

EXERCISES ON PARALLELOGRAMS (*continued*).

10. In the diagonal AC of the parallelogram ABCD two points X, Y are taken so that AX = CY; and XB, YB, XD, YD are joined. Show that XBYD is a parallelogram.

11. If in the sides of a parallelogram ABCD four points P, Q, R, S are taken in order, one in each side, so that AP = BQ = CR = DS, prove that the figure PQRS is a parallelogram.

12. A is the vertex of an isosceles △ ABC, and BA is produced to any point D. CX is drawn parallel to BA to meet the bisector of the exterior ∠ DAC at X. Show that ABCX is a parallelogram.

13. The diagonals of a rectangle divide the figure into two congruent triangles: is the diagonal, therefore, an axis of symmetry? About what two lines is a rectangle symmetrical? [See Def. p. 65.]

14. Is there any axis about which an oblique parallelogram is symmetrical? Give reasons for your answer.

15. In a quadrilateral ABCD, AB = AD and CB = CD; but the sides are not all equal. Which of the diagonals (if either) is an axis of symmetry?

16. Two quadrilaterals ABCD, EFGH have the sides AB, BC, CD, DA equal respectively to the sides EF, FG, GH, HE, and have also the angle BAD equal to the angle FEH. Show that the figures are congruent.

17. Using the result of Theorem 20, show how a straight line of given length may be bisected by means of a set square construction.

18. Two straight lines AB, BC meet at B. In AB take any point X, and suppose XY is drawn perpendicular to AB meeting BC at Y. Draw YZ perpendicular to XY and equal to BY. Show that if BZ is joined it bisects the ∠ ABC.

19. Two bars AP, BQ are pivoted at fixed points A and B. A straight rule PQ is pivoted to the bars at P and Q, the distances between the pivots A, P and B, Q being equal, and PQ being equal to AB. Show that in all positions of the system the rule PQ is parallel to AB; and that the bars AP, BQ rotate through equal angles.

20. ABC and DEF are two triangles such that AB, BC are respectively equal to and parallel to DE, EF; show that AC is equal and parallel to DF.

21. ABCD is a quadrilateral in which AB is parallel to DC, and AD equal but not parallel to BC; show that

(i) the $\angle A$ + the $\angle C = 180°$ = the $\angle B$ + the $\angle D$;

(ii) the diagonal AC = the diagonal BD;

(iii) the quadrilateral is *symmetrical* about the straight line joining the middle points of AB and DC.

22. AP, BQ are straight rods of equal length, turning at equal rates (both clockwise) about two fixed pivots A and B respectively. If the rods start parallel but pointing in opposite senses, show that

(i) they will always be parallel; (ii) the line joining PQ will always pass through a certain fixed point.

(Miscellaneous Questions and Exercises.)

23. State the properties of a triangle relating to

(i) the sum of its interior angles; (ii) the sum of its exterior angles.

What property corresponds to (i) in a polygon of *n* sides? With what other figures does a triangle share the property (ii)?

24. Calculate the angles of a triangle ABC, having given:

$$\text{int. } \angle A = \tfrac{5}{7} \text{ of ext } \angle A; \; 3B = 4C.$$

25. A yacht sailing due East changes her course successively by 63°, by 78°, by 119°, and by 64°, with a view to sailing round an island. What further change must be made to set her once more on an Easterly course?

26. If the sum of the interior angles of a rectilineal figure is equal to the sum of the exterior angles, how many sides has it, and why?

27. Draw, using your protractor, any five-sided figure ABCDE, in which

$$\angle B = 110°, \quad \angle C = 115°, \quad \angle D = 93°, \quad \angle E = 152°.$$

Verify by a construction with ruler and compasses that AE is parallel to BC, and account theoretically for this fact.

28. A and B are two fixed points, and two straight lines AP, BQ, unlimited towards P and Q, are pivoted at A and B. AP, starting from the direction AB, turns about A clockwise at the uniform rate of $7\frac{1}{2}°$ a second; and BQ, starting simultaneously from the direction BA, turns about B counter-clockwise at the rate of $3\frac{3}{4}°$ a second.

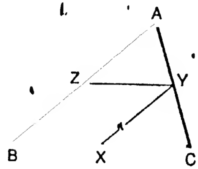
(i) How many seconds will elapse before AP and BQ are parallel?

(ii) Find graphically and by calculation the angle between AP and BQ twelve seconds from the start.

(iii) At what rate does this angle decrease?

THEOREM 22.

The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.



In the $\triangle$ ABC, let Z be the middle point of the side AB, and let ZY be drawn parallel to BC to meet AC in Y.

To prove that AY = YC.

Proof. Suppose YX to be drawn parallel to AB, meeting BC at X,

Then BXYZ is a parallelogram.

$\therefore$ XY = the opposite side BZ

= ZA. *Given.*

Now because AC cuts the parallels XY, BA,

$\therefore$ the $\angle$ XYC = the corresponding $\angle$ BAC.

Also because AC cuts the parallels ZY, BC,

$\therefore$ the $\angle$ AYZ = the corresponding $\angle$ ACB.

Hence in the $\triangle^s$ XYC, ZAY,

because $\begin{cases} \text{the } \angle \text{ XYC} = \text{the } \angle \text{ ZAY;} \\ \text{and the } \angle \text{ YCX} = \text{the } \angle \text{ AYZ} & \textit{Proved.} \\ \text{and XY} = \text{ZA,} & \textit{Proved.} \end{cases}$

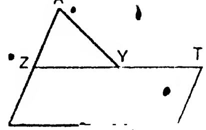
$\therefore$ the triangles are congruent;

so that AY = YC. Q.E.D.

[For Exercises on Theorems 22, 23, see pp. 90, 91.]

Theorem 23.

The straight line joining the middle points of two sides of a triangle is parallel to the third side, and equal to half of it.



In the $\triangle$ ABC, let Z and Y be the middle points of the sides AB, AC.

To prove that ZY *is parallel to* BC, *and equal to half of it.*

Suppose ZY produced to T, and YT made equal to ZY.

Join CT.

Proof. In the $\triangle$s CYT, AYZ,

because { CY = AY, *Given.*
and YT = YZ by construction,
and the $\angle$CYT = the vertically opposite $\angle$AYZ,

$\therefore$ the triangles are congruent;

$\therefore$ CT = AZ

= BZ, *Given.*

and the $\angle$YCT = the $\angle$YAZ.

But these angles are alternate angles made by AC meeting CT, AB;

$\therefore$ CT is parallel to AB.

Hence CT, BZ are parallel and equal;

so that BCTZ is a parallelogram:

$\therefore$ ZT and BC are parallel and equal; that is, ZY is parallel to BC.

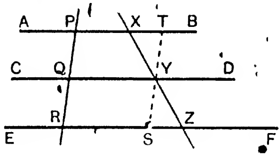
But ZY is half of ZT: $\therefore$ ZY is half of BC. Q.E.D.

Note. This Theorem might be derived from Theorem 22. Thus: If through Z a parallel were drawn to BC, it would bisect AC (Th. 22); that is, it would pass through Y. Hence the line joining Z and Y, and the line through Z parallel to BC, are one and the same.

That is, ZY is parallel to BC.

THEOREM 24.

If three or more parallel straight lines make equal intercepts on any transversal, they make equal intercepts on any other transversal.



Let the parallels AB, CD, EF make equal intercepts PQ, QR on the transversal PQR; and let XY, YZ be the corresponding intercepts made on any other transversal XYZ.

To prove that XY = YZ.

Through Y let TYS be drawn parallel to PR and cutting AB, EF at T and S.

Proof. Since AB, CD, EF are parallel, and also PR, TS,

∴ the figures PY and QS are parallelograms.

∴ TY = PQ, and YS = QR.

But PQ = QR, by hypothesis,

∴ TY = YS.

Now in the △s XYT, ZYS,

because { the ∠XTY = the alternate ∠ZSY, and the ∠XYT = the vertically opposite ∠ZYS, and TY = YS,

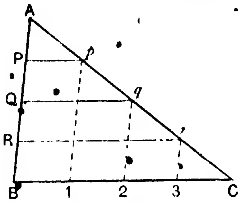
∴ the triangles are congruent;

so that XY = YZ. Q.E.D.

Conversely, *if* AB, EF *are parallel, and* Q, Y *are the middle points of the intercepts* PR, XZ, *show that* QY *is parallel to* AB *and* EF.

[For TY = YS, from the congruent △s XTY, ZSY. Also TS = PR, from the par^{m} PS. Hence the halves of these are equal, that is, YS = QR. Since YS, QR are equal and parallel, the fig. QS is a par^{m}, so that QY is par^{l} to RS.]

COROLLARY. *In a triangle ABC, if a set of lines Pp, Qq, Rr, ..., drawn parallel to one side BC, divide another side AB into equal parts, they also divide the third side AC into equal parts.*



The lengths of the parallels Pp, Qq, Rr, ..., may thus be expressed in terms of the base BC.

Consider the figure in which AB is divided into four equal parts.

Through p, q, and r let p1, q2, r3 be drawn par^l to AB

Then, by Theorem 24, these par^{ls} divide BC into *four* equal parts, of which Pp evidently contains *one*, Qq *two*, and Rr *three*.

In other words,

$$Pp = \frac{1}{4} BC; \qquad Qq = \frac{2}{4} BC; \qquad Rr = \frac{3}{4} BC.$$

Similarly if the given par^{ls} divide AB into n equal parts,

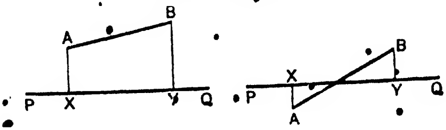
$$Pp = \frac{1}{n} BC, \qquad Qq = \frac{2}{n} BC, \qquad Rr = \frac{3}{n} BC, \text{ and so on.}$$

*** *Problem* 6, *p.* 106, *should now be worked.*

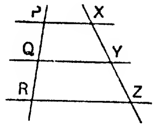
[For Exercises, see pp. 90, 91.]

DEFINITION.

If from the extremities of a straight line AB perpendiculars AX, BY are drawn to a straight line PQ of indefinite length, then XY is said to be the **orthogonal projection** of AB on PQ.



EXERCISES ON PARALLELS AND PARALLELOGRAMS



1. (i) Prove Theorem 24 by joining PZ and applying Theorem 22 in turn to the $\triangle^s$ PRZ, ZPX.

(ii) Prove Theorem 24 by drawing XM and YN parallel to PR to meet QY and RZ at M and N, and proving the $\triangle^s$ XMY, YNZ are congruent.

2. In a $\triangle$ ABC a point P is taken in AB so that AP = $\frac{1}{3}$AB, and through P a parallel PM is drawn to BC. Prove that

(i) AM = $\frac{1}{3}$AC; (ii) PM = $\frac{1}{3}$BC.

3. **Show that the three straight lines which join the middle points of the sides of a triangle, divide it into four congruent triangles.**

4. From the result of the last Exercise show how a triangle may be drawn *with set squares*, if the positions of the middle points of the three sides are known.

5. **Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.**

6. ABCD is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC: show that BX and DY trisect the diagonal AC.

7. **If the middle points of adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram.**

8. Show that the straight lines which join the middle points of opposite sides of a quadrilateral, bisect one another.

9. AB is a given straight line bisected at O; and AX, BY are perpendiculars drawn to any other straight line. Prove that OX = OY.

10. From two points A and B, and from O the mid-point between them, perpendiculars AP, BQ, OX are drawn to a straight line CD;

(i) if A and B are on the *same* side of CD, prove that
$$OX = \tfrac{1}{2}(AP + BQ);$$

(ii) if A and B are on opposite sides of CD, prove that
$$OX = \tfrac{1}{2}(AP \sim BQ).$$

[The symbol $\sim$ is used for "the difference between."]

11. In the figure of Ex. 1, PX, QY, RZ being given parallel, and Q being the mid-point of PR, prove that
$$QY = \tfrac{1}{2}(PX + RZ).$$

QY is said to be the **Arithmetic Mean** between PX and RZ.

12. In the trapezium ABCD, AB and DC are the parallel sides, and X, Y the mid-points of the oblique sides. Prove that

(i) XY is parallel to AB, DC.

(ii) $XY = \frac{1}{2}(AB + DC)$.

13. If X and Y are the mid-points of AD, BC, the oblique sides of the trapezium ABCD, show that XY bisects both diagonals AC and BD.

14. OX and OY are two straight lines, and along OX five points 1, 2, 3, 4, 5 are marked at equal distances Through these points parallels are drawn in any direction to meet OY. Measure the lengths of these parallels: take their average, and compare it with the length of the *third* parallel. Prove *geometrically* that the 3rd parallel is the mean of all five.

State the corresponding theorem for any odd number $(2n+1)$ of parallels so drawn

15. From the angular points of a parallelogram perpendiculars are drawn to any straight line which is outside the parallelogram: show that the sum of the perpendiculars drawn from one pair of opposite angles is equal to the sum of those drawn from the other pair.

[Draw the diagonals, and from their point of intersection suppose a perpendicular drawn to the given straight line]

16 The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides is equal to the perpendicular drawn from either extremity of the base to the opposite side.

[It follows that the sum of the distances of *any* point in the base of an isosceles triangle from the equal sides is **constant**, that is, the same whatever point in the base is taken]

How would this property be modified if the given point were taken in the base *produced* ?

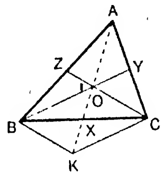
17. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the perpendicular drawn from any one of the angular points to the opposite side, and is therefore constant.

18. **Equal and parallel lines have equal projections on any other straight line.**

THE MEDIANS OF A TRIANGLE.

DEFINITION. The straight line joining a vertex of a triangle to the middle point of the opposite side is called a **median**.

The medians of a triangle meet at a point which is a point of trisection of each median.



In the $\triangle$ ABC, let BY and CZ be medians cutting at O.

Join AO, and produce it to meet BC at X.

To prove that AX *is the third median.*

Through C draw CK par^l to BY, to meet AX produced at K; join BK.

Proof. In the $\triangle$ AKC, because AY = YC, and YO is par^l to CK,

$\therefore$ AO = OK. *Theor.* 22.

Again, in the $\triangle$ ABK, because AZ = ZB, and AO = OK,

$\therefore$ ZO is par^l to BK; *Theor.* 23.

that is, OC is par^l to BK.

$\therefore$ the figure BKCO is a par^m.

And since the diagonals BC, OK bisect one another,

$\therefore$ X is the middle point of BC.

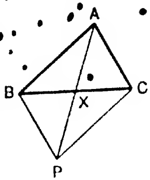
That is, AX is the third median; so that the three medians meet at O.

Also, since AO = OK, and OX = one-half of OK,

$\therefore$ OX = one-half of AO = one-third of AX.

DEFINITION. The point of intersection of the medians is called the **centroid** of the triangle.

EXERCISES ON THE MEDIANS OF A TRIANGLE.



1. In the △ ABC, the median AX bisects BC and is produced to P, so that XP = AX. Join BP, CP, and prove that ABPC is a parallelogram.

2. In the △ ABC, show that the sum of the sides AB, AC is greater than twice the median AX.

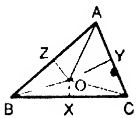
3. Prove that the perimeter of a triangle is greater than the sum of the medians. [Apply Ex. 2 to each pair of sides in turn.]

4. Construct a △ ABC, having given the lengths of AB, AC and the median AX.

[Here the problem is to place AB, AX, AC so that B, X, C are in a straight line, and X is the mid point of BC.

Consider the fig. of Ex. 1, and begin by constructing the △ ABP.]

The following examples should be worked out in full.



5. ABC is a △, and X, Y, Z the mid points of its sides. YO, ZO are perp. to AC, AB, meeting at O. Join OX, *and prove OX perp. to* BC.

Join OA, OB, OC.

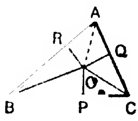
Prove

(i) the △s AZO, BZO congruent (Theor. 9)

(ii) the △s AYO, CYO congruent (Theor. 9);
Deduce OA = OB = OC.

(iii) the △s OXB, OXC congruent (Theor. 13).
Deduce ∠OXB = ∠OXC = a right angle.

Hence OX is perp. to BC, and the three perpendicular bisectors of the sides meet at O, which is equidistant from A, B, C.



6. ABC is a △, and BO, CO bisect the ∠s B and C meeting at O. Join AO, *and prove that* AO *bisects the* ∠ A.

From O draw OP, OQ, OR perp. to BC, CA, AB.

Prove

(i) the △s OPB, ORB congruent (Theor. 10)

(ii) the △s OPC, OQC congruent (Theor. 10)
Deduce OP = OQ = OR.

(iii) the △s OQA, ORA congruent (Theor. 14).
Deduce ∠OAQ = ∠OAR.

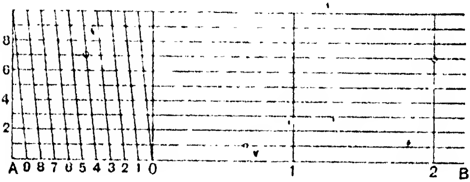
Hence OA bisects the ∠ A, and the three angular bisectors meet at O, which is equidistant from BC, CA, AB.

NOTE. Two or more straight lines which meet in a point are said to be concurrent.

Diagonal Scales.

We here explain, as an important application of Theorem 24, the construction of Diagonal Scales. Their use is to make linear measurements of greater accuracy than is possible with a graduated ruler. The scale now described is *a Decimal Diagonal Scale to show Inches, Tenths, and Hundredths.*

A straight line AB is divided (from A) into inches, and the points of division marked 0, 1, 2, . . . The primary division 0A is subdivided into *tenths*, these secondary divisions being numbered (from 0) 1, 2, 3, ... 9. We may now read on AB *inches* and *tenths* of an inch.



In order to read *hundredths*, ten lines are taken at any equal intervals parallel to AB, and perpendiculars are drawn through 0, 1, 2, ...

The primary (or inch) division corresponding to 0A on the tenth parallel is now subdivided into *ten* equal parts, and diagonal lines are drawn, as in the diagram,

joining 0 to the *first* point of subdivision on the 10th parallel,
" 1 to the *second* " " " "
" 2 to the *third* " " " "

and so on.

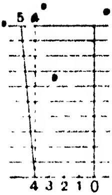
The scale is now complete, and its use is shown in the following example.

Example. *To take from the scale a length of 2·47 inches.*

(i) Place one point of the dividers at 2 in AB, and extend them till the other point reaches 4 in the subdivided inch 0A. We have now 2·4 inches in the dividers.

(ii) To get the remaining 7 *hundredths*, move the right-hand point up the perpendicular through 2 till it reaches the 7th parallel. Then extend the dividers till the left point reaches the diagonal 4 also on the 7th parallel. We have now 2·47 inches in the dividers.

REASON FOR THE ABOVE PROCESS.



The first step needs no explanation. The reason of the second is found in the Corollary of Theorem 2[illegible].

Joining the point 4 to the corresponding point on the tenth parallel, we have a triangle 4,4,5, of which one side 4,4 is divided into ten equal parts by a set of lines parallel to the base 4,5.

Therefore the lengths of the parallels between 4,4 and the diagonal 4,5 are $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, ... of the base, which is ·1 inch.

Hence these lengths are respectively

·01, ·02, ·03, ... of 1 inch.

Similarly, by this scale the length of a given straight line may be measured to the nearest hundredth of an inch.

NOTE. The subdivision of a diagonal scale need not be *decimal.* For instance we might construct a diagonal scale to read centimetres, millimetres, and *quarters* of a millimetre; in which case we should take *four* parallels to the line AB.

EXERCISES ON LINEAR MEASUREMENTS.

1. Draw straight lines whose lengths are 1·25 inches, 2·72 inches, 3·08 inches.

2. Draw a line 2·68 inches long, and measure its length in centimetres and the nearest millimetre.

3. Draw a line 5·7 cm. in length, and measure it in inches (to the nearest hundredth). Check your result by calculation, given that 1 cm. = 0·3937 inch.

4. Find by measurement the equivalent of 3·15 inches in centimetres and millimetres. Hence calculate (correct to two decimal places) the value of 1 cm. in inches.

5. Draw lines 2·9 cm. and 6·2 cm. in length, and measure them in inches. Use each equivalent to find the value of 1 inch in centimetres and millimetres, and take the average of your results.

6. A distance of 100 miles is represented on a map by 1 inch. Draw lines to represent distances of 336 miles and 408 miles.

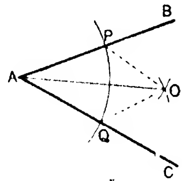
7. If 1 inch on a map represents 1 kilometre, draw lines to represent 850 metres, 2980 metres, and 1010 metres.

8. A plan is drawn to the scale of 1 inch to 100 links. Measure in centimetres and millimetres a line representing 417 links.

SECTION I. CONSTRUCTIONS

PROBLEM 1.

To bisect a given angle.



Let BAC be the given angle to be bisected.

Construction. With centre A, and any radius, draw an arc of a circle cutting AB, AC at P and Q.

With centres P and Q, and radius PQ, draw two arcs cutting at O. Join AO.

Then the ∠ BAC is bisected by AO.

Proof. Join PO, QO.

In the △s APO, AQO,

because { AP = AQ, being radii of a circle,
PO = QO, " " equal circles,
and AO is common;

∴ the triangles are congruent; *Theor.* 13.

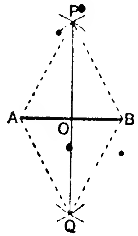
so that the ∠ PAO = the ∠ QAO;

that is, the ∠ BAC is bisected by AO.

NOTE. PQ has been taken as the radius of the arcs drawn from the centres P and Q, and the intersection of these arcs determines the point O. *Any* radius, however, may be used instead of PQ, provided that it is great enough to secure the intersection of the arcs.

PROBLEM 2.

To bisect a given straight line.



Let AB be the line to be bisected.

Construction. With centre A, and radius AB, draw two arcs, one on each side of AB.

With centre B, and radius BA, draw two arcs, one on each side of AB, cutting the first arcs at P and Q.

Join PQ cutting AB at O.

Then AB is bisected at O.

Outline of Proof. Join AP, AQ, BP, BQ.

First show that the $\triangle$s APQ, BPQ are congruent; *Theor.* 13.

so that the $\angle$APQ = the $\angle$BPQ.

Then show that the $\triangle$s APO, BPO are congruent; *Theor.* 9.

so that AO = OB;

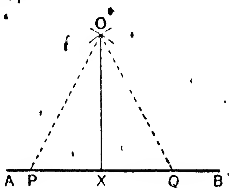
that is, AB is bisected at O.

NOTES. (i) AB was taken as the radius of the arcs drawn from the centres A and B, but *any* radius may be used provided that it is great enough to secure the intersection of the arcs which determine the points P and Q.

(ii) From the congruence of the $\triangle$s APO, BPO it follows that the $\angle$AOP = the $\angle$BOP. As these are adjacent angles, PQ *bisects* AB *at right angles*.

PROBLEM 3.

To draw a straight line perpendicular to a given straight line at a given point in it.



Let AB be the straight line, and X the given point in it.

Construction. With centre X cut off from AB any two equal parts XP, XQ.

With centres P and Q, and radius PQ, draw two arcs cutting at O.

Join XO

Then XO is perp. to AB.

Outline of Proof. Join OP, OQ.

Show that the △s OXP, OXQ are congruent; *Theor.* 13.

so that the ∠OXP = the ∠OXQ.

And these being adjacent angles, each is a right angle;

that is, XO is perp. to AB.

Obs. If the point X is near one end of AB, one or other of the alternative constructions on the next page should be used.

PROBLEM 3. SECOND METHOD.

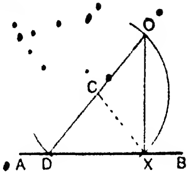
Construction. Take any point C outside AB.

With centre C, and radius CX, draw a circle cutting AB at D.

Join DC, and produce it to meet the circumference of the circle at O.

Join XO.

Then XO is perp. to AB



Proof. Join CX.

Because CO = CX, ∴ the ∠CXO = the ∠COX;
and because CD = CX, ∴ the ∠CXD = the ∠CDX.
∴ the whole ∠DXO = the ∠XOD + the ∠XDO
= $\frac{1}{2}$ of 180°
= 90°.

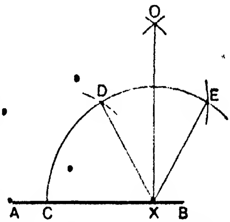
∴ XO is perp. to AB.

PROBLEM 3. THIRD METHOD.

Construction. With centre X and any radius, draw the arc CDE, cutting AB at C.

With centre C, and with the same radius, draw an arc, cutting the first arc at D.

With centre D, and with the same radius, draw an arc cutting the first arc at E.



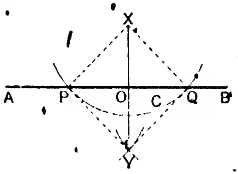
Bisect the ∠DXE by XO. *Prob.* 1.

Then XO is perp. to AB.

Proof. Each of the ∠s CXD, DXE may be proved to be 60°;
and the ∠DXO is half of the ∠DXE;
∴ the ∠CXO is 90°.
That is, XO is perp. to AB.

PROBLEM 1.

To draw a straight line perpendicular to a given straight line from a given point outside it.



Let AB be the given straight line and X the given point outside it.

Construction. Take any point C on the side of AB remote from X.

With centre X, and radius XC, draw an arc to cut AB at P and Q.

With centres P and Q, and radius PX, draw arcs, cutting at Y, on the side of AB opposite to X.

Join XY cutting AB at O.

Then XO is perp. to AB.

Outline of Proof. Join PX, QX, PY, QY.

First show that the $\triangle$s PXY, QXY are congruent; *Theor.* 13.

so that the $\angle$PXY = the $\angle$QXY.

Then show that the $\triangle$s PXO, QXO are congruent. *Theor.* 9.

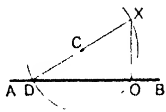
so that the $\angle$XOP = the $\angle$XOQ.

And these being adjacent angles, each is a right angle,

that is, XO is perp. to AB.

Obs. When the point X is nearly opposite one end of AB, one or other of the alternative constructions given below should be used.

Problem 4. Second Method



Construction. Take any point D in AB. Join DX, and bisect it at C.

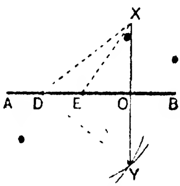
With centre C, and radius CX, draw a circle cutting AB at D and O.

Join XO.

Then XO is perp. to AB.

For, as in Problem 3, Second Method, the $\angle$ XOD is a right angle.

Problem 4. Third Method



Construction. Take any two points D and E in AB.

With centre D, and radius DX, draw an arc of a circle, on the side of AB opposite to X.

With centre E, and radius EX, draw another arc cutting the former at Y.

Join XY, cutting AB at O.

Then XO is perp. to AB.

(i) Prove the $\triangle$s XDE, YDE congruent by Theor. 13, so that the $\angle$ XDE = the $\angle$ YDE.

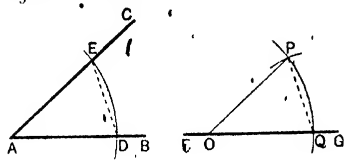
(ii) Hence prove the $\triangle$s XDO, YDO congruent by Theor. 9, so that the adjacent $\angle$s DOX, DOY are equal.

That is, XO is perp. to AB.

*** *The constructions of Problems 3 and 4 are not usually followed in practical applications. Perpendiculars as well as parallels are more readily drawn by the aid of set squares.*

Problem 5.

At a given point in a given straight line to make an angle equal to a given angle.



Let BAC be the given angle, and FG the given straight line; and let O be the point at which an angle is to be made equal to the ∠BAC.

Construction. With centre A, and with any radius, draw an arc cutting AB and AC at D and E.

With centre O, and with the same radius, draw an arc cutting FG at Q.

With centre Q, and with radius DE, draw an arc cutting the former arc at P.

Join OP.

Then POQ is the required angle.

Outline of Proof. Join ED, PQ.

Then prove the △s POQ, EAD congruent by Theor. 13.

It will follow that ∠POQ = ∠EAD.

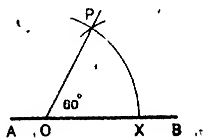
CONSTRUCTIONS WITH RULER AND COMPASSES OF ANGLES OF 60°, 30°, 45°.

1. *At a point O in a straight line AB to construct an angle of 60°.*

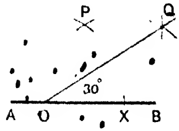
From O as centre, with any convenient radius, draw an arc cutting AB at X.

From centre X, with *the same* radius, cut the first arc at P.

Join OP; and without formal proof say why the ∠BOP = 60°.



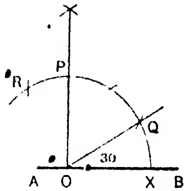
2. *At a point* O *in a straight line* AB *to construct an angle of* 30°.



(i) This may be done by constructing an angle of 60° as above, and bisecting it by Prob. 1. The construction may be arranged as in the figure, and needs no further explanation

(ii) Or, An angle of 30° may be constructed by trisecting a right angle, thus

With centre O, and any radius OX, draw the arc XR.

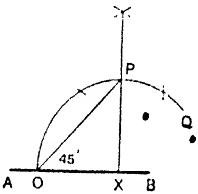


At O draw OP perp. to AB (by Prob. 3, *Third* Method), meeting the arc XR at P.

From centre P, with radius OX, cut the arc PX at Q.

Join OQ; and say why the ∠BOQ 30°.

3. *At a point* O *in a straight line* AB *to construct an angle of* 45°.



Take any point X in AB, and from centre X, with radius XO, draw the arc OQ.

Hence (by Prob. 3, *Third* Method) draw at X a perp. to AB, cutting the arc OQ at P.

Join OP; and say why the ∠BOP = 45°.

EXERCISES.

1. Construct (with ruler and compasses only) angles of (i) 120°, (ii) 135°, (iii) 15°, (iv) $22\frac{1}{2}$°, (v) $112\frac{1}{2}$°, (vi) $157\frac{1}{2}$°.

2. Through a given point P draw (with set squares) straight lines making the following angles with a given straight line AB: (i) 60°, (ii) 30°, (iii) 45°.

3. Each angle of an equilateral triangle is 60°. Say why; and use this property to prove that in a triangle ABC *right-angled at* C, if the hypotenuse AB = 2BC, then the ∠B = 60°.

EXERCISES.

(General Theoretical Constructions.)

N.B. *Give the construction in each case. A full formal proof is required only when specially asked for; but in every example the reason for the construction should be briefly explained. Draw parallels and perpendiculars with set squares unless otherwise directed.*

1. If PQ bisects a given line AB at right angles, *prove that* any point in PQ is equidistant from A and B.

Hence show how to find in a straight line XY a point which is equidistant from A and B.

When is this impossible?

2. Four points A, B, C, D are given in position. Find by construction a point P such that PA = PB, and PC = PD.

When is this impossible?

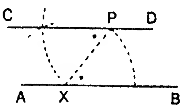
3. Find a point that is equidistant from three given points A, B, C. In what positions of A, B, C would your construction fail?

4. P is any point on the bisector of the angle BAC. Prove that perpendiculars drawn from P to the arms of the angle are equal.

Hence show how to find in a straight line XY a point equidistant from two intersecting lines AB, AC. When is this impossible?

5. **Show how to draw through a given point P a straight line parallel to a given straight line AB with ruler and compasses only.**

[In AB take any point X, and join PX. At the point P in XP make the ∠ XPC equal to the ∠ PXB and alternate to it. (Problem 5.)]



6. From a given point P draw a straight line PQ, making with a given straight line AB an angle equal to a given angle L.

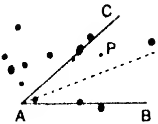
7. O is the middle point of a straight line AB. *Prove* that the points A and B are equidistant from any straight line through O.

Hence through a given point P draw a straight line such that the perpendiculars drawn to it from two points A and B may be equal.

Is this always possible?

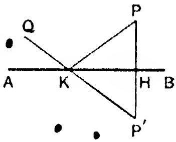
8. AB, AC are two intersecting straight lines, and P is a given point within the ∠BAC. Required to draw through P a straight line meeting AB, AC at X and Y, and cutting off equal intercepts AX and AY.

[Bisect the ∠BAC, and complete the construction.]



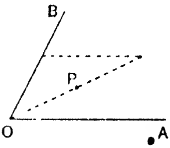
9. From two given points P and Q on the same side of a straight line AB, it is required to draw two lines which meet in AB and make equal angles with it.

[Devise a construction with the help of the adjoining figure, and prove that PK, QK are the required lines.]



10. P is a given point within the angle AOB. Draw through P a straight line terminated by OA and OB, and bisected at P.

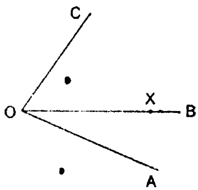
[Take a hint from the adjoining figure, and complete the construction. Supply a proof. See Theorem 20.]



11. OA, OB, OC are three straight lines meeting at O. Draw a transversal terminated by OA and OC, and bisected by OB.

[Complete the construction from the adjoining figure, working from any point X in OB; and give a proof. See Theorem 20.]

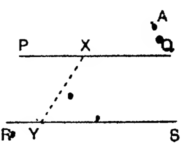
How many such transversals could be drawn?



12. Through a given point A draw a straight line so that the part intercepted between two given parallels may be of given length.

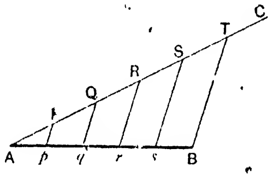
When does this problem admit of two solutions? When of only one? And when is it impossible?

[Take any point X in PQ; and from centre X and the given length as radius, cut RS at Y. Complete the construction. Proof by Theorem 19.]



Problem 6.

To divide a given straight line into any number of equal parts.



Let AB be the given straight line, and suppose it is required to divide it into *five* equal parts.

Construction. From A draw AC, a straight line of unlimited length, making any angle with AB.

From AC mark off *five* equal parts of *any* length, AP, PQ, QR, RS, ST.

Join TB, and through P, Q, R, S draw (with set squares) par[ls] to TB, meeting AB in p, q, r, s.

Then since the par[ls] Pp, Qq, Rr, Ss, TB cut off five equal parts from AT, they also cut off five equal parts from AB. (Theorem 21. Cor.)

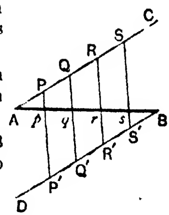
Second Method.

From A draw AC at any angle with AB, and on it mark off *four* equal parts AP, PQ, QR, RS, of any length.

From B draw BD par[l] to AC, and on it mark off BS′, S′R′, R′Q′, Q′P′, each equal to the parts marked on AC.

Join PP′, QQ′, RR′, SS′ meeting AB in p, q, r, s. Then AB is divided into five equal parts at these points.

[Prove by Theorems 21 and 24.]



NOTE. By means of Prob. 6 we may divide a straight line AB in any given ratio (provided that the ratio can be expressed in terms of whole numbers).

For suppose the given ratio is 3 : 2. In this case divide AB into *five* equal parts (as in the figure) of which Ax contains 3, and xB contains 2. Then evidently AB is divided at x in the ratio 3 : 2.

Or, more generally, if the given ratio is $m : n$, we must divide AB into $m+n$ equal parts; then if AX contains m of these parts, XB must contain n. So that AB is divided at X in the ratio $m : n$.

EXERCISES

1. Draw a line of length 7·2 cm., and divide it into *five* equal parts. Measure one of the parts, and verify numerically.

2. Draw (with diagonal scale) a line of length 3·61 inches, and divide it into *seven* equal parts. Measure one part, and verify numerically.

3. Draw a straight line 6·8 cm. long in the ratio 3 : 4. Measure each part, and verify by reckoning.

4. Draw a line of length 3·82 inches, and divide it in the ratio 4 : 5. Measure each part, and verify numerically.

5. Draw a line 6·7 cm. long, and divide it into *five* equal parts. Measure one of the parts in inches (to the nearest hundredth), and verify your work by calculation. [1 cm. = 0·3937 inch.]

6. Find to the nearest hundredth of an inch the length of a line which will represent 42·500 kilometres in a map drawn to the scale of 1 centimetre to 5 kilometres.

7. On a map of France drawn to the scale 1 inch to 35 miles, the distance from Paris to Calais is represented by 4·2 inches. Find the distance accurately in miles, and approximately in kilometres, and express the scale in metric measure. [1 km. = $\frac{5}{8}$ mile, nearly.]

8. Draw a diagonal scale, 2 centimetres to represent 1 yard, showing yards, feet, and inches.

On the Construction of Triangles.

In the cases hitherto considered [Introduction, pp. 30-36] we have seen that

(i) In order to construct a triangle *three* parts have been given, for example:

(*a*) two sides and the included angle;

(*b*) one side and the angles at its ends;

(*c*) the three sides.

(ii) A triangle cannot be constructed from *any* three given parts, for example, if the *three angles* are given (and no side), the problem is indeterminate that is, an unlimited number of triangles may be drawn having the given parts.

The reason of this is that the *data* in this case are not *independent*, for if *two* angles are given, the third is known necessarily.

(iii) That even when the three parts are independent, they may have to be in some way restricted or related before a triangle can be drawn from them For example:

(*a*) if one side and the angles at its ends are given, the sum of angles must be less than 180°;

(*b*) if the three sides are given, then the sum of any two must be greater than the third.

(iv) Lastly, if the given parts are sufficient on construction to fix the size and shape of the triangle, then all triangles having the same parts must be congruent

We shall give instances of the construction of triangles where the data are not necessarily all *sides* or *angles* of the triangle. To such cases the same general observations apply: namely.

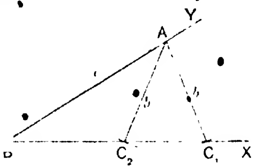
Three independent data are required; and if the given conditions are possible, and sufficient to determine the triangle, then all triangles fulfilling these conditions are congruent.

[The simplest cases of the construction of triangles, already dealt with in the Introduction, need no further explanation, but a few more examples will be given for practice]

PROBLEM 7.

To construct a triangle ABC *having given two sides* (b, c) *and an angle* (B) *opposite to one of them*

For example, construct a triangle having given $b = 3.6$ cm., $c = 6.4$ cm., B = 30°.



Construction. Take any straight line BX, and at B make the ∠ XBY (or B) equal to 30°.

From BY cut off BA equal to c that is, to 6.4 cm. (The diagram is drawn to half this scale.)

From centre A, and with radius b (or 3.6 cm.) draw an arc of a circle.

This arc, in the present case, will be found to cut BX in two points C_1 and C_2, both on the same side of B, so that both triangles ABC_1, ABC_2 satisfy the given conditions.

This double solution is known as the **Ambiguous Case,** and will occur when b is less than c but greater than the perpendicular from A on BX.

[Compare the Ambiguous Case in the congruence of triangles explained on p. 71.]

EXAMPLE.

Draw figures to illustrate the nature and number of solutions in the following cases:

(i) When b is greater than c. (Say $b = 8.6$ cm.)

(ii) When b is equal to c.

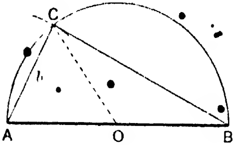
(iii) When b is equal to the perpendicular from A on BX, in this case 3.2 cm.

(iv) When b is less than this perpendicular.

PROBLEM 8.

To construct a RIGHT-ANGLED *triangle* ABC *having given the hypotenuse* (c) *and one side* (b).

For example, construct a triangle given C = 90°, $c = 7{\cdot}6$ cm., $b = 3{\cdot}6$ cm.



Construction. Make AB equal to 7·6 cm. for the hypotenuse, and bisect it at O. With centre O, and radius OA, draw a semi-circle.

With centre A, and radius b (that is, 3·6 cm.), draw an arc cutting the semi-circumference at C.

Join AC, BC.

Then ABC is the required triangle.

Proof. Join OC.

Because OA = OC;

∴ the ∠OCA = the ∠OAC.

And because OB = OC;

∴ the ∠OCB = the ∠OBC.

∴ the whole ∠ACB = the ∠OAC + the ∠OBC

= ½ of 180° *Theor.* 7.

= 90°.

Or thus: Draw AC of the given length b, and at C draw CB perp. to AC and of indefinite length.

With centre A, and radius equal to the hypotenuse c, cut CB at B. Join AB. Then ABC is the required triangle.

Observe that two *congruent* triangles may arise from this construction; for CB may be cut on either side of C. Compare Theorem 14.

ON THE CONSTRUCTION OF TRIANGLES.

(Draw parallels and perpendiculars with set squares, and angles with protractor, unless otherwise directed. In each case begin by making a rough free-hand sketch.)

1. Draw a triangle ABC in which $b = 5·3$ cm., $c = 6·8$ cm., and $B = 45°$. Can more than one triangle have these parts? If so, measure and compare the angles C.

2. Construct a triangle ABC right angled at C, having $c = 4·0"$, and $a = 2·4"$.

3. Draw a triangle ABC in which $B = 30°$, $c = 6·0$ cm., $b = 9·0$ cm., and show that this can be done by two different constructions.

4. Draw an isosceles triangle on a base of 6·0 cm., and having an altitude of 9·3 cm. Give reasons for your construction.

5. On a base of length 2·4" construct an isosceles triangle having a vertical angle of 48°. Describe your construction, and give a reason for it.

6. Construct triangles from the following data. In each case say how many solutions there are, and state the reason.

(i)	$b = 2·8"$,	$c = 2·3"$,	$B = 70°$
(ii)	$a = 5·8$ cm.,	$b = 6·9$ cm.,	$A = 52°$
(iii)	$b = 10·0$ cm.,	$c = 7·4$ cm.,	$C = 48°$
(iv)	$a = 4·2"$,	$c = 2·6"$,	$A = 107°$

7. From the foot of a lighthouse L two ships A and B, which are 600 yards apart, are observed in directions S.W. and 15° East of South respectively. At the same time B is observed from A in a S.E. direction. Draw a plan (scale 1" to 200 yds.), and find by measurement the distance of the lighthouse from each ship.

8. Draw a right-angled triangle, given that the hypotenuse $c = 10·6$ cm. and one side $a = 5·6$ cm. Measure the third side b, and find the value of $\sqrt{c^2 - a^2}$. Compare the two results.

9. Draw with compasses and ruler a triangle PQR right-angled at R, given that the hypotenuse $= 3·2"$, and the $\angle Q = 60°$. Without formal proof, give a reason for your construction.

10. Two straight roads, which cross at right angles at A, are carried over a straight canal by bridges at B and C. The distance between the bridges is 461 yards, and the distance from the crossing A to the bridge B is 261 yards. Draw a plan, and by measurement of it ascertain the distance from A to C.

CONSTRUCTION OF TRIANGLES (*continued*).

11. Construct a triangle, having given the following parts: B = 34°, b = 5.5 cm., c = 8.5 cm. Show that there are two solutions. Measure the two values of C, and show that these are supplementary.

12. In a triangle ABC, the angle A = 50°, and b = 6.5 cm. Illustrate by figures the cases which arise in constructing the triangle, when (i) a = 7 cm. (ii) a = 6 cm. (iii) a = 5 cm. (iv) a = 4 cm.

13. Draw a △ ABC right-angled at B, and having the hypotenuse AC = 2AB.

Hence show how to construct an isosceles triangle in which each of the equal sides = twice the altitude. What will be its vertical angle?

14. (i) Construct an isosceles triangle having a vertical angle of 50°, and the perpendicular from the vertex on the base of length 8.5 cm.

(ii) Draw (with ruler and compasses only) an equilateral triangle in which the perpendicular from one vertex to the opposite side is 8 cm.

[Where the perpendicular from the vertex on the base is given, the perpendicular should be drawn first; then the base must lie in a straight line of indefinite length at right angles to the perpendicular.]

15. Construct a right-angled triangle having given the length of one side containing the right angle, and the perpendicular from the right angle to the hypotenuse.

For example: let the side = 6.5 cm., and the perpendicular = 5.4 cm.

Show that there are two solutions, and that the triangles obtained are congruent.

16. Construct a △ ABC in which the perpendicular from A to BC (or BC produced) is 5.0 cm., and the sides AB, AC are 5.8 cm. and 8.0 cm. respectively.

Show that there are *four* solutions, which may be grouped in two pairs of congruent triangles.

17. Construct a △ ABC, having given ∠B = 40°, ∠C = 76°, and the perpendicular from A to BC = 7.0 cm.

18. Construct a triangle ABC (without protractor) having given two angles B and C and the side b.

19. From B the foot of a flagstaff AB a horizontal line is drawn passing two points C and D which are 27 feet apart. The angles BCA and BDA are 65° and 40° respectively. Represent this on a diagram (scale 1 cm. to 10 ft.), and find by measurement the approximate height of the flagstaff.

[First construct the △ ACD.]

20. From P, the top of a lighthouse PQ, two boats A and B are seen at anchor in a line due south of the lighthouse. It is known that PQ = 126 ft., ∠PAQ = 57°, ∠PBQ = 33°; hence draw a plan in which 1″ represents 100 ft., and find by measurement the distance between A and B, to the nearest foot.

21. Construct a *right angled* triangle, having given the length of the hypotenuse c, and the sum of the remaining sides a and b.

For example, c = 5·3 cm., and $a + b$ = 7·3 cm.

Show that in this case there are *two* solutions, and *prove* that the triangles obtained are congruent.

[Draw PB of length equal to $(a + b)$. At P make an angle BPR = 45°, then complete the construction.]

22. Construct a triangle having given the perimeter and the angles at the base. For example, $a + b + c$ = 12 cm., B = 70°, C = 80°.

[Draw PQ of length equal to $(a + b + c)$. From P draw PA making ∠QPA = ½B. From Q draw QA making ∠PQA = ½C. Complete the construction, and explain the reason for it.]

23. Construct a triangle ABC from the following data:

a = 6·5 cm., $b + c$ = 10 cm., and B = 60°.

Test by measuring the lengths of b and c.

24. Construct a triangle ABC from the following data:

a = 7 cm., $c - b$ = 1 cm., and B = 55°.

Measure the lengths of b and c.

25. Construct an *isosceles* triangle ABC, having given the perimeter and the length of the perpendicular from the vertex A to the base BC.

Suppose $a + b + c$ = 3·8″, and perpendicular = 1·2″.

What relation must exist between the data in order that the construction may be possible?

26. Construct a triangle ABC, having given BC = 3·0″, ∠B = 70°, and the median AX = 2·5″.

27. Construct a triangle ABC, having given the length of BC, also the lengths of the medians BY, CZ.

[See fig., p. 92. Take two-thirds of BY, CZ, and begin by constructing the △OBC.]

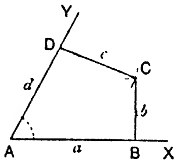
28. Construct a triangle having given the lengths of its three medians. [See fig., p. 92; and begin by constructing the △OKC.]

THE CONSTRUCTION OF QUADRILATERALS.

It has been shown that the shape and size of a triangle are completely determined when the lengths of its three sides are given. A quadrilateral, however, is not completely determined by the lengths of its four sides. From what follows it will appear that *five* independent data are required to construct a quadrilateral.

PROBLEM 9.

To construct a quadrilateral ABCD, *given the lengths of the four sides* (a, b, c, d), *and one angle* A (*as shown in the figure*).



Suppose $a = 6·6$ cm., $b = 3·0$ cm., $c = 4·2$ cm., $d = 5·1$ cm.; and $\angle$A $= 59°$.

Construction. Take any straight line AX, and cut off from it AB equal to a.

Make the $\angle$A to contain 59°, as given.

From AY cut off AD equal to d.

With centre D, and radius c, draw an arc of a circle.

With centre B, and radius b, draw another arc to cut the former at C.

Join DC, BC.

Then ABCD is the required quadrilateral; for by construction the sides are equal to a, b, c, d, and the $\angle$A is of the required size.

NOTE. Observe that the arcs drawn from centres B and D will meet (if the data are possible) at *two* points. If B and D are joined to the other point of intersection (C′), notice that the $\angle$BC′D will be reflex. Examine the case when $b + c =$ the diagonal BD.

EXERCISES.

1. Draw a *parallelogram* ABCD (with ruler and compasses), having given that the adjacent sides AB, AD are 3·1" and 1·7" respectively, and the included ∠A = 60°.

Explain why these data amount to *five* independent conditions.

2. Draw a square on a side of 2·5" (using set squares). Measure a diagonal; and test your construction by working out (correct to 2 decimal places) the formula *diag. of square = side* × $\sqrt{2}$.

3. With ruler and compasses draw a square on a diagonal of 3·0".

4. Draw a rhombus each of whose sides is equal to a given straight line PQ, which is also to be one diagonal of the figure.

5. Draw a parallelogram ABCD, having given that

(i) the two adjacent sides AB, AD are 6·5 cm. and 4·5 cm., and the diagonal AC = 8·5 cm.

(ii) AB = 5·5 cm., and the diagonals AC, BD are 8 cm. and 6 cm.

6. Draw a rectangle in which one side = 1·8", and one diagonal = 3·5".

7. Construct a parallelogram ABCD (with set squares and protractor), having given

(i) AB = 7 cm., the ∠A = 55°, and diagonal BD = 6 cm.

(ii) AB = 7 cm., the ∠A = 55°, and diagonal AC = 11·5 cm.

Examine the number of solutions in each case.

8. In a quadrilateral ABCD,

AB = 5·6 cm., BC = 2·5 cm., CD = 4·0 cm., and DA = 3·3 cm.

Show that the shape of the quadrilateral is not settled by these data.

Draw the quadrilateral when (i) A = 30°, (ii) A = 60°. Why does the construction fail when A = 100°?

Find graphically the least value of A for which the construction fails.

9. Show how to construct a quadrilateral, having given the lengths of the four sides and of one diagonal. What conditions must hold among the data in order that the problem may be possible?

Illustrate your method by constructing a quadrilateral ABCD, when

(i) AB = 3·0", BC = 1·7", CD = 2·5", DA = 2·8"; diagonal BD = 2·6".

(ii) AB = 3·6 cm., BC = 7·7 cm., CD = 6·8 cm., DA = 5·1 cm., and the diagonal AC = 8·5 cm.

10. Construct a quadrilateral ABCD, having given three sides and the two diagonals; namely,

AB = 8·2 cm., AD = 7·4 cm., BC = 5·0 cm.,

AC = 8·4 cm., BD = 9·0 cm.

[For further Exercises on Quadrilaterals, see pp. 118, 148, 153.]

MISCELLANEOUS EXAMPLES ON SECTION I

1. ABCD is a parallelogram, AB is produced to E so that BE = AB; prove that ED bisects BC.

2. The bisectors of the angles B and C of a triangle ABC meet at X. Prove that $\angle$BXC is obtuse.

3. ABC and ADB are two triangles on the same side of AB, such that AC = BD, and AD = BC. If AD and BC intersect in P, prove that the triangle APB is isosceles.

4. ABCD is a square, and E is the mid-point of DC. A circle with centre E cuts AD in P and BC in Q. Prove that the triangles PED and QEC are congruent. Prove also that BP and AQ are equal.

5. ABC is a triangle and the side BC is produced to D. If the bisector of the angle ACD is parallel to AB, prove that the triangle is isosceles.

6. The interior angles of a quadrilateral are 40°, 100°, 64°, and 156°. The bisectors of the exterior angles form a new quadrilateral. Find the interior angles of this new quadrilateral.

7. On two adjacent sides AB, AD of a parallelogram ABCD, squares ABEF, ADGH are described externally. Prove that HF is equal to the diagonal AC of the parallelogram.

8. The angle BCA of the triangle ABC is twice the angle CAB, and its bisector meets AB at N. The perpendicular from N to AC meets CB produced at F. Prove that the triangle CFA is isosceles.

9. O is a point inside a triangle ABC such that $\angle OBC = \frac{2}{3}\angle ABC$, $\angle OCA = \frac{2}{3}\angle BCA$. Prove that $\angle BOC = 60° + \frac{2}{3}\angle CAB$.

10. The diagonals of a parallelogram are 6·8 cm. and 5·6 cm. long, and are inclined at an angle of 67°. Construct the figure and measure its angles.

11. OAB is a triangle in which OA = OB, and is such that if the bisector of the angle A cuts OB at K, then OK = KA. Prove that ten triangles congruent to OAB can be fitted round O to form a regular ten-sided figure.

12. AB and A′B′ are two equal straight lines. Find a point C such that the triangles ABC, A′B′C are congruent.

13. The two angles A and B of a triangle ABC are bisected by straight lines meeting in O. Prove that the angle AOB exceeds half the angle C by a right angle.

14. In a triangle ABC the side AB is greater than the side AC, and the bisectors of the exterior angles at B and C meet at D. Prove that DC is greater than DB.

15. ABCD is a parallelogram. BC is produced to P, making CP = CD, and BA is produced to Q, making AQ = AD. Prove that P, D, and Q are collinear, that is, in the same straight line.

16. From the vertex A of a right angled triangle a perpendicular is drawn to the hypotenuse BC. The bisector of the angle B meets this perpendicular in P and meets AC in Q. Prove that the triangle APQ is isosceles.

17. A, B, C, D are four points such that AC = AD, and BC = BD. Prove that any point on the line AB is equidistant from C and D.

18. Show that the straight line joining the middle points of the diagonals of a trapezium is parallel to the two parallel sides of the trapezium and equal in length to half their difference.

19. A, B, C are three points on a level plain; it is required to run a line through C parallel to AB. How would you do this if you are provided with a measuring tape and some pegs? Justify your method.

20. P, O, Q are three points in this order on a straight line. An acute angle POA is drawn on one side of PQ, and an angle QOB equal to three times POA is drawn on the other side of PQ, OB being equal to OA. If AB cuts PQ at C, prove that OC = CA.

21. If ABCD is a square, and AL, BM are drawn perpendicular to any line through D, and AN is drawn perpendicular to BM or BM produced, prove that ALMN is a square.

22. D is a point on BC, a side of an equilateral triangle ABC. An equilateral triangle CDE is described on CD, the vertices A and E being on opposite sides of BC. Prove that AD = BE, and if AD produced meets BE in F, prove that the angle BFD = 60°.

23. AB and AC are two straight lines representing railway lines, and P is a village lying between them, which is known to be in a straight line with and equidistant from two stations one on each line; find the position of the stations.

24. Draw an irregular pentagon, having each of its angles greater than a right angle; produce the sides of the pentagon so as to form a five-pointed star, the points of the star being the intersections of alternate sides. Prove that the sum of the angles at the points of the star is equal to two right angles.

25. On opposite sides AB, CD of a square ABCD equilateral triangles ABE, CDF are described outside the square. Prove that DE and BF are parallel.

26. ABCD is a rectangle whose diagonal AC is twice the side AB; CD is produced to P so that DP = CD. If O is the middle point of AC, prove that OP is perpendicular to AC and equal to AD.

27. From any point P in the base of an isosceles triangle ABC, straight lines are drawn parallel to the sides meeting them in D and E. Prove that the sum of PD and PE is the same for all positions of P.

28. The sum of the angles B and C of a triangle is 105°, and the angle A exceeds the angle C by 40°. If BC = 2″, construct the triangle.

29. In an acute-angled triangle ABC in which AB is greater than AC, the lines drawn from B, C perpendicular to the opposite sides intersect in O. Prove that OB is greater than OC.

30. ABCD is a parallelogram. Equal straight lines CE, DF are drawn in opposite directions (outside the parallelogram) at right angles to BC and DA respectively. Prove that EF bisects CD.

31. In a triangle ABC the side AB is greater than the side AC; the angle BAC is bisected by a line meeting BC at D. Show that the angle ADB is obtuse.

32. A quadrilateral ABCD has the adjacent sides AB, AD equal, and also the opposite angles B, D equal. Prove that its diagonals are perpendicular to each other.

33. Construct a quadrilateral ABCD in which the side AB = 2″, ∠ABC = 120°, the diagonal AC = 3″, and the sides CD and DA each = 2½″. If DE is the perpendicular from D to AC, prove that it bisects AC and the ∠D. Show by measurement that DE = AB.

34. In a triangle ABC the angle A is 60°. Prove that CB + ½BA is greater than CA.

35. ABCDE, ABPQRS are respectively a regular pentagon and a regular hexagon described on the same side of a given line AB. Join PC. Calculate the size of the angles QPC, PCD.

36. Find a point P in the base BC of a triangle ABC so that CP is equal to the perpendicular from P on AB.

37. Construct a quadrilateral ABCD from the following data: AB = 5.2 cm.; AD = 10.8 cm.; ∠BAD = 80°; the diagonals AC, BD are equal and include an angle of 64°. Explain your construction.

38. Make an accurate drawing of a quadrilateral ABCD, having the angles B, C, D respectively 2, 2½, and 2½ times the angle A, the side AB 3″ in length, and the opposite side 1″. Justify your construction.

39. Construct, explaining each step, a quadrilateral ABCD with AB parallel to DC, ∠BAD = 135°, ∠ACD = 60°, DC = 2″, DB = 2½″.

40. Construct a quadrilateral ABCD with AB = 1″, AC = 2″, angle ABC = 90°, and the triangle ADC equilateral. Prove that the ∠BCD = 90°.

SECTION II THEOREMS.

On Areas.

Definitions.

1. The **altitude** (or **height**) of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side.

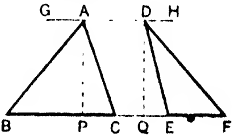
2. The **altitude** (or **height**) of a triangle with reference to a given side as base, is the perpendicular distance of the opposite vertex from the base.

Obs. It is clear that *parallelograms or triangles which are between the same parallels have the same altitude.*

For let AP and DQ be the altitudes of the ∆s ABC, DEF, which are between the same parallels BF, GH.

Then the fig. APQD is evidently a rectangle;

∴ AP = DQ.



Conversely. *Parallelograms and triangles which have the same altitude may be so placed as to be between the same parallels.*

Note. When two parallelograms, or two triangles, are described as being *between the same parallels*, it is always understood that the figures are so placed that the bases lie on one of the parallels, and that the opposite sides of the parallelograms (or the vertices of the triangles) lie on the other parallel.

3. The **area** of a figure is the amount of surface contained within its bounding lines.

4. A **square inch** is an area equivalent to that of a square drawn on a side one inch in length.

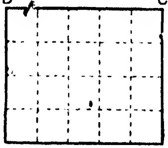
5. Similarly a square centimetre is an area equivalent to that of a square drawn on a side one centimetre in length.

The terms *square yard, square foot, square metre* are to be understood in the same sense.

6. Thus the **unit of area** is the area of a square on a side of unit length.

THEOREM 25.

Area of a rectangle. The number of **units of area** contained in a rectangle is the product of the number of **linear units** in its length and breadth.



Let ABCD represent a rectangle whose length AB is 5 feet, and whose breadth AD is 4 feet.

Divide AB into 5 equal parts, and BC into 4 equal parts, and through the points of division of each line draw parallels to the other.

The rectangle ABCD is now divided into compartments, each of which represents one square foot.

Now there are four rows, each containing 5 squares,

$\therefore$ the rectangle contains 5×4 square feet.

Similarly, if a denotes the *number* of linear units in the length, and b the *number* of linear units in the breadth,

then the rectangle contains **ab** *units of* **area.**

And if each side of a square contains a linear units,

the square contains **a²** *units of* **area.**

These statements may be thus abridged

the area of a rectangle = *length* × *breadth*............(i),

the area of a square = (*side*)^2 ....(ii)

In other words,

The area of a rectangle is measured by the product of the measures of two adjacent sides.

NOTATION.

The rectangle ABCD is said to be **contained** by AB, AD; for these adjacent sides fix its size and shape.

The area of a rectangle whose adjacent sides are AB, AD is denoted by *rect* AB, AD, or simply AB × AD.

The area of a square drawn on the side AB is denoted by *sq. on* AB, or AB^2.

INFERENCES.

It is evident that

(i) *Rectangles which have two adjacent sides of one equal to two adjacent sides of the other, each to each, are congruent, and therefore have equal areas.*

(ii) *If rectangles have equal areas and if one side of one is equal to one side of the other, then the adjacent sides are also equal.*

For example, if rectangles have equal areas and equal lengths, they have also equal breadths.

(iii) *Squares on equal sides are congruent, and therefore have equal areas.*

(iv) *If squares have equal areas they stand on equal sides.*

EXERCISES.

(On Tables of Length and Area.)

1. Draw a figure to show *why*

(i) 1 sq. yard = 3^2 sq. feet

(ii) 1 sq. cm = 10^2 sq. mm

2. Draw a figure to show that the square on a straight line is four times the square on half the line.

3. Use squared paper to show that the square on 1″ = 10^2 times the square on 0·1″.

4. If 1″ represents 5 miles, what does an area of 6 square inches represent?

EXTENSION OF THEOREM 25.

The proof of Theorem 25 on page 120 supposes that the length and breadth of the given rectangle are expressed by *whole numbers*; but the formula holds good when the length and breadth are fractional.

This may be illustrated thus:

Suppose the length and breadth are 3·2 cm. and 2·4 cm.; we shall show that the area is $(3{\cdot}2 \times 2{\cdot}4)$ sq. cm.

For length = 3·2 cm. = 32 mm.
breadth = 2·4 cm. = 24 mm.

$$\therefore \text{ area} = (32 \times 24) \text{ sq. mm.} = \frac{32 \times 24}{10^2} \text{ sq. cm.}$$
$$= (3{\cdot}2 \times 2{\cdot}4) \text{ sq. cm.}$$

NOTE. Observe that the rule here holds good because the length and breadth, though *fractional* in terms of centimetres, may be expressed by *whole numbers* in terms of a lower unit, in this case millimetres; and we may thus divide the rectangle into squares as illustrated on p. 120. In this case the length and breadth are said to be **commensurable**.

We shall refer later to the case of **incommensurable** lines (or other magnitudes), that is to say, such as cannot be divided into parts of which each contains an exact whole number. For example, lines whose measures are 1 and $\sqrt{2}$ furnish a familiar instance of **incommensurables**. In such cases it will appear that, by choosing a sufficiently small unit, lines which are theoretically incommensurable may be expressed by whole numbers *to any required degree of accuracy*, that is, to any number of significant digits.

EXERCISES.

(On the Area of a Rectangle.)

Draw on squared paper the rectangles of which the length (a) and breadth (b) are given below. Calculate the areas, and verify by the actual counting of squares.

1. $a = 0{\cdot}8''$, $b = 3{\cdot}5''$.
2. $a = 2{\cdot}5''$, $b = 1{\cdot}4''$.
3. $a = 2{\cdot}2''$, $b = 1{\cdot}5''$.
4. $a = 1{\cdot}6''$, $b = 2{\cdot}1''$.

Calculate the areas of the rectangles in which

5. $a = 18$ metres, $b = 11$ metres.
6. $a = 7$ ft., $b = 72$ in.
7. $a = 2{\cdot}5$ km., $b = 4$ metres.
8. $a = \frac{1}{2}$ mile, $b = 1$ inch.

9. Find the length of a rectangle whose area is 3·9 sq. in., and breadth 1·5″. Draw the rectangle on squared paper; and verify your work by counting the squares.

10. (i) When you treble the length of a rectangle without altering its breadth, how many times do you multiply the area?

(ii) When you treble both length and breadth, how many times do you multiply the area?

Draw a figure to illustrate your answers, and state a general rule.

11. In a plan of a rectangular garden the length and breadth are 3·6″ and 2·5″, one inch standing for 10 yards. Find the area of the garden.

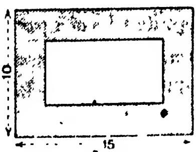
If the area is increased by 300 sq. yds., the breadth remaining the same, what will the new length be? And how many inches will represent it on your plan?

12. Find the area of a rectangular inclosure of which a plan (scale 1 cm. to 20 metres) measures 6·5 cm. by 4·5 cm.

13. The area of a rectangle is 1440 sq. yds. If in a plan the sides of the rectangle are 3·2 cm. and 4·5 cm., on what scale is the plan drawn?

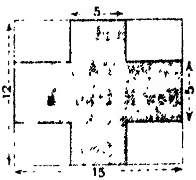
Calculate the areas represented by the shaded parts of the following plans. The dimensions are marked in feet.

14.



Width of shaded border uniform 2½ ft.

15.



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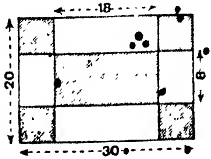
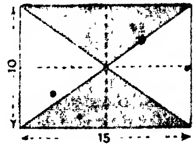
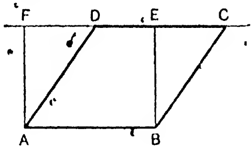


17.



THEOREM 26.

The area of a parallelogram is equal to the area of the rectangle on the same base and between the same parallels.



Let ABCD be a par^{m} and ABEF the rectangle on the same base AB and between the same par^{ls} AB, FC.

To prove that the par^{m} ABCD *the rect* ABEF *in area.*

Proof. Because FC meets the par^{ls} AD, BC,

∴ the ∠ADF = the corresponding ∠BCE.

In the △^{s} ADF, BCE,

because { the ∠ADF = the ∠BCE, *Proved.*
the ∠AFD = the ∠BEC, being rt. angles ;
and AD BC, being opposite sides of a par^{m}.

∴ the △^{s} ADF, BCE are congruent and equal in area

Now if from the whole fig. ABCF the △ ADF is taken, the remainder is the par^{m} ABCD ;

And if from the same fig. ABCF the △ BCE is taken, the remainder is the rect. ABEF ;

∴ these remainders are equal ;

that is, the par^{m} ABCD = the rect. ABEF. Q.E.D.

EXAMPLE.

In the above diagram the sides FE, DC overlap. Draw diagrams in which (i) these sides do not overlap ; (ii) the ends E and D coincide.

Go through the proof with these diagrams, and ascertain if it applies to them without change.

COROLLARY 1. *The* **area of a parallelogram** *is measured by the product of the measures of its base and its altitude.*

In the figure of Theor. 26 BE is the altitude of the par^{m} ABCD.

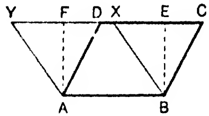
Now if AB, BE contain respectively a and p units of length, the rect ABEF contains ap units of area.

And, by Theor. 26, area of par^{m} ABCD = area of rect. ABEF
= ap units of area.

This result may be thus abridged :

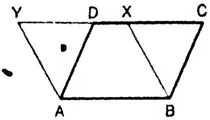
area of par^{m} = base × altitude.

COROLLARY 2. *Parallelograms on the same base and between the same parallels (hence of the same altitude) are equal in area*



For if the par^{ms} ABCD, ABXY are on the same base AB and between the same par^{s} AB, YC ; and if ABEF is the rectangle on the base AB and of the same altitude AF as the par^{ms} ;

then the par^{m} ABCD = the par^{m} ABXY ; since, by Theor 26, each is equal in area to the rect. ABEF.



NOTE. This important Corollary may be proved *directly*, that is, without reference to the equivalent rectangle on the same base and between the same par^{ls} [see the adjoining figure]. The pupil should as an exercise go fully through this form of proof, which closely follows that of Theor 26.

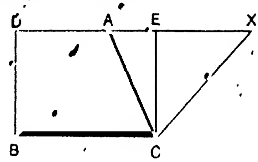
COROLLARY 3. Since the area of a parallelogram depends only on its base and altitude, it follows that

Parallelograms on equal bases and of equal altitudes are equal in area.

[For Exercises, see pp. 128-130]

Theorem 27.

The area of a triangle is equal to one-half of the area of the rectangle on the same base and between the same parallels.



Let ABC be a triangle, and DBCE the rectangle on the same base BC and between the same par^{ls} BC, DE.

To prove that the △ ABC = *one-half of the rect.* DBCE *in area.*

Let CX be the straight line through C par^{l} to BA, meeting DE produced at X.

Proof. Then ABCX is a par^{m}, and its diagonal AC bisects it;

∴ the △ ABC = one-half of the par^{m} ABCX

= one-half of the rect. DBCE, on the same base and between the same parallels.

Q.E.D.

Corollary 1. *The* **area of a triangle** *is measured by one-half of the product of the measures of its base and its altitude.*

In the above figure CE = the altitude of the △ ABC.

Now if BC, CE contain respectively a and p units of length, then the rect. DBCE contains ap units of area.

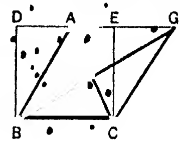
And, by Theor. 27, area of △ ABC = one-half of rect. DBCE

$= \frac{1}{2} . ap$ units of area

This result may be abridged thus:

area of triangle $= \frac{1}{2}$. *base* × *altitude.*

COROLLARY 2 *Triangles on the same base and between the same parallels (i.e. of the same altitude) are equal in area.*



For if the $\triangle$s ABC, GBC are on the same base BC and between the same par[ls] BC, DG; and if DBCE is the rectangle on the base BC and of the same altitude BD as the triangles;

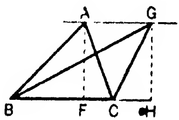
then the $\triangle$ ABC = one-half of the rect. DBCE;

also the $\triangle$ GBC = one-half of the rect. DBCE;

$\therefore$ the $\triangle$ ABC = the $\triangle$ GBC in area.

Similarly, *triangles on equal bases and of equal altitudes are equal in area.*

COROLLARY 3. *If two triangles are equal in area, and stand on the same base and on the same side of it, they are between the same parallels.*



For instance, if the $\triangle$s ABC, GBC are equal in area, and stand on the same base BC, it is concluded that BC and AG are par[l].

For if AF and GH are the altitudes of the triangles,

then the $\triangle$ ABC = one-half of the rect. BC, AF;

also the $\triangle$ GBC = one-half of the rect. BC, GH;

$\therefore$ BC $\times$ AF = BC $\times$ GH;

$\therefore$ the altitude AF = the altitude GH.

Also AF and GH are par[l];

hence AG and FH, that is BC, are par[l]. *Theor.* 21.

Similarly, *triangles of the same area, standing on equal bases in the same straight line, are between the same parallels.*

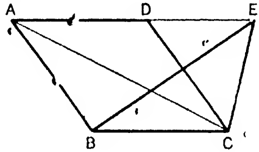
And generally, since the *area of a triangle* = $\frac{1}{2}$. *base* $\times$ *altitude*, it follows that

(i) *triangles of equal area and equal bases have equal altitudes;*

(ii) *triangles of equal area and equal altitudes have equal bases.*

THEOREM 28.

If a parallelogram and a triangle stand on the same base and between the same parallels, the area of the parallelogram is double that of the triangle.



Let the par^m ABCD and the △ EBC stand on the same base BC and between the same par^{ls} BC, AE.

To prove that the par^m ABCD is double of the △ EBC in area.

Join AC

Proof. Since the △s ABC, EBC are on the same base BC and between the same par^{ls} BC, AE,

∴ the △ ABC = the △ EBC in area

And since the par^m ABCD is bisected by its diagonal AC,

∴ the par^m ABCD is double of the △ ABC,

∴ the par^m ABCD is double of the △ EBC in area.

Q.E.D.

EXERCISES.

(Mensuration of Parallelograms and Triangles)

1. Find the area of parallelograms in which
   (i) the base = 5·5 cm., and the height = 4 cm.
   (ii) the base = 2·4", and the height 1·5".

2. Draw a parallelogram ABCD having given AB = 2½", AD = 1¼", and the ∠A = 65°. Draw and measure the perpendicular from D on AB, and hence calculate the approximate area. Why *approximate* ?

Again calculate the area from the length of AD and the perpendicular on it from B. Obtain the average of the two results.

3. In the parallelogram ABCD the following lengths are given:

AB = a units; AD = b units;

the perp. distance between AB and DC = p units;
the perp. distance between AD and BC = q units;

prove that $ap = bq$.

4. The area of a parallelogram ABCD is 4·2 sq. in., and the base AB is 2·8". Find the height. If AD = 2", draw the parallelogram.

5. Each side of a rhombus is 2", and its area is 3·86 sq. in. Calculate an altitude, and hence draw the rhombus.

6. Calculate the areas of the triangles in which

(i) the base = 24 ft., the height = 15 ft.

(ii) the base = 4·8", the height = 3·5".

7. Draw triangles from the following data. In each case draw and measure the altitude with reference to a given side as base; hence calculate the approximate area.

(i) a = 8·5 cm., b = 6·8 cm., c = 4·0 cm.

(ii) b = 5·0 cm., c = 6·8 cm., A = 65°.

(iii) a = 6·5 cm., B = 52°, C = 76°.

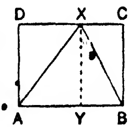
8. ABC is a triangle *right-angled* at C; show that its area = ½BC × CA.
Given a = 6 cm., b = 5 cm., calculate the area.

Draw the triangle and measure the hypotenuse c; draw and measure the perpendicular from C on the hypotenuse; hence calculate the approximate area. Compare your result with the calculated value.

9. In a triangle it is given that the area = 10·4 sq. cm., and the base = 1·6 cm. Calculate the altitude. Can you draw the triangle from these data?

THEORETICAL EXERCISES ON AREAS.

(Parallelograms and Triangles.)



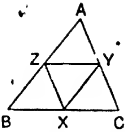
1. (i) Prove *directly from the adjoining figure* that the △ XAB = one-half of the rectangle ABCD.

(ii) Draw the corresponding figure when X is in DC produced; and prove that in this case also the △ XAB = one-half of the rect. ABCD.

THEORETICAL EXERCISES ON AREAS.

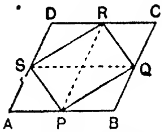
(Parallelograms and Triangles continued.)

2. If X, Y, Z are the mid-points of the sides of the ∆ ABC, point out three parallelograms (Theor. 23) in the figure, and prove that they are equal in area.



Are these parallelograms congruent? Distinguish between the cases, when the ∆ ABC is (i) equilateral, (ii) isosceles, (iii) scalene.

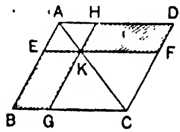
3. If P, Q, R, S, are the mid-points of the sides of the par^m ABCD, prove that



(i) PR is par^l to AD, BC; and SQ par^l to AB, DC;

(ii) the fig PQRS is a par^m, and that its area is half that of the par^m ABCD.

4. ABCD is a par^m, and through any point K in the diagonal AC par^ls EF, HG are drawn, as in the adjoining figure. Why are the following pairs of triangles congruent?



∆s ADC, CBA; ∆s AHK, KEA; ∆s KFC, CGK.

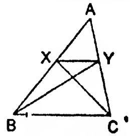
Hence prove that

(i) the shaded par^ms HF, EG are equal in area,

(ii) par^m ED = par^m BH;

(iii) ∆ AKB = ∆ AKD.

5. ABC is a triangle and XY is drawn parallel to the base BC, cutting the other sides at X and Y. Join BY and CX; and show that



(i) the ∆ XBC = the ∆ YBC in area;

(ii) the ∆ BXY = the ∆ CXY „ „

(iii) the ∆ ABY = the ∆ ACX „ „

If BY and CX cut at K, show that

(iv) the ∆ BKX = the ∆ CKY in area.

6. Show that a median of a triangle divides it into two parts of equal area.

How would you divide a triangle into *three* equal parts by straight lines drawn from its vertex?

7. Prove that a parallelogram is divided by its diagonals into four triangles of equal area.

8. ABC is a triangle whose base BC is bisected at X. If Y is any point in the median AX, show that

the ∆ ABY = the ∆ ACY in area.

9. ABCD is a parallelogram, and BP, DQ are the perpendiculars from B and D on the diagonal AC.

Show that BP = DQ.

Hence if X is any point in AC, or AC produced, prove that

(i) the △ADX = the △ABX in area;

(ii) the △CDX = the △CBX ,, ,,

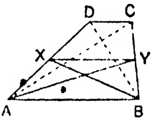
10. In a triangle ABC, if X, Y are the mid-points of the sides AB, AC, prove (without assuming Theor. 23) that the △s XBY, XCY are equal in area. Hence show that XY and BC are parallel.

11. ABCD is a trapezium having DC par^l to AB; and X, Y are the mid-points of AD, BC.

Why is the △AXB one-half of the △ADB in area?

Of what triangle is the △AYB one-half?

Why may we conclude that the △AXB = the △AYB in area? And hence that XY is par^l to AB?



12. ABCD is a parallelogram, and X, Y are the middle points of the sides AD, BC; if Z is any point in XY, or XY produced, show that the area of the triangle AZB is one quarter of the parallelogram ABCD.

13. If ABCD is a parallelogram, and X, Y any points in DC and AD respectively: show that the triangles AXB, BYC are equal in area.

14. ABCD is a parallelogram, and P is any point within it; show that the sum of the triangles PAB, PCD is equal to half the parallelogram.

15. In a △ABC, X, Y are the mid-points of the sides AB, AC, and BY, CX intersect at K. Join AK, and prove that

the △BKC = the △AKB = the △AKC in area.

16. In AB, a side of the △ABC, any point X is taken, and CX is joined. Through A a straight line is drawn par^l to XC and meeting BC produced at Y; and XY is joined. Prove that the △XBY = the △ABC in area.

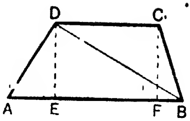
17. If the middle points of the sides of a quadrilateral are joined in order, prove that the *parallelogram* so formed (see Ex. 7, p. 99) is half the quadrilateral.

18. ABC is a triangle, and R, Q the middle points of the sides AB, AC; show that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral AQXR.

FURTHER EXAMPLES ON AREAS.

I. QUADRILATERALS.

(i) *To find the area of a* **trapezium.**



Let ABCD be a trapezium, having the sides AB, CD parallel. Join BD, and from C and D draw perpendiculars CF, DE to AB.

Let the parallel sides AB, CD measure a and b units of length, and let the height CF contain h units.

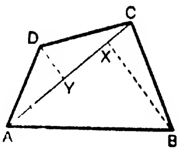
Then the area of ABCD = $\triangle$ ABD + $\triangle$ DBC

$$= \frac{1}{2}AB \cdot DE + \frac{1}{2}DC \cdot CF$$

$$= \frac{1}{2}ah + \frac{1}{2}bh = \frac{h}{2}(a+b).$$

That is, *area of trapezium* $= \frac{1}{2}$ *height* $\times$ *(sum of parallel sides).*

(ii) *To find the area of* **any quadrilateral.**



Let ABCD be any quadrilateral. Draw a diagonal AC, and from B and D draw perpendiculars BX, DY to AC. These perpendiculars are called **offsets.**

Let AC contain d units of length, and BX, DY p and q units respectively:

then area of the quad' ABCD = $\triangle$ ABC + $\triangle$ ADC

$$= \frac{1}{2}AC \cdot BX + \frac{1}{2}AC \cdot DY$$

$$= \frac{1}{2}dp + \frac{1}{2}dq = \frac{1}{2}d(p+q).$$

That is, *area of quadrilateral* $= \frac{1}{2}$ *diagonal* $\times$ *(sum of offsets).*

EXERCISES.

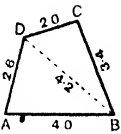
(Numerical and Graphical.)

1. Find the area of the trapezium in which the two parallel sides are 4·7" and 3·3", and the height 1·5".

2. In a quadrilateral ABCD, the diagonal AC = 17 feet, and the offsets from it to B and D are 11 feet and 9 feet. Find the area.

3. Draw a quadrilateral ABCD from the adjoining rough plan, the dimensions being given in inches.

Draw and measure the offsets to A and C from the diagonal BD, and hence calculate the area of the quadrilateral.



4. Draw a trapezium ABCD from the following data: AB and CD are the parallel sides; AB = 4"; AD = BC = 2"; the ∠A = the ∠B = 60°.

Make any necessary measurements, and calculate the area.

(Theoretical.)

5. In a quadrilateral ABCD one diagonal BD passes through X the mid point of the other diagonal AC. Prove that BD bisects the quadrilateral.

6. ABCD is a parallelogram of which the diagonals AC, BD cut *at right angles*. It is required to show that

area of such a par^m = ½ *(product of diagonals).*

(i) Deduce this result from the formula

area of any quadrilateral = ½ *diagonal* × *(sum of offsets).*

(ii) Obtain the same result direct from the following construction: Through A and C draw parallels to BD, through B and D draw parallels to AC; and let XYZV be the *rectangle* so formed.

7. Given the lengths of the diagonals of a quadrilateral, and the angle between them, prove that the area is the same wherever they intersect.

8. In the trapezium ABCD, AB is parallel to DC; and X is the middle point of BC. Through X draw PQ parallel to AD to meet AB and DC produced at P and Q. Then prove

(i) trapezium ABCD = par^m APQD.

(ii) trapezium ABCD = twice the △ AXD.

THEORETICAL EXERCISES ON AREAS CONTINUED.

(Triangles and Quadrilaterals.)

9. Prove that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals.

Hence show how a parallelogram ABCD may be bisected by a straight line drawn

(i) through a given point P;

(ii) perpendicular to the side AB;

(iii) parallel to a given line QR.

10. If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides *supplementary*, show that the triangles are equal *in area*. Can such triangles ever be congruent?

11. Show that a trapezium is bisected in area by the straight line which joins the middle points of its *parallel* sides.

[Join the ends of one of the parallel sides to the mid-point of the other.]

12. Through A and C, two opposite vertices of a quadrilateral ABCD, parallels are drawn to the diagonal BD; and through B and D parallels are drawn to AC.

Prove that the figure thus formed is a *parallelogram*, and that its area is double that of the quadrilateral ABCD.

13. Use the completed figure of the last exercise to show that the area of the quadrilateral ABCD = the area of the triangle having two sides equal to the diagonals AC, BD, and the included angle equal to either of the angles between the diagonals.

14. Two triangles of equal area stand on the same base but on opposite sides of it: show that the straight line joining their vertices is bisected by the base, or by the base produced.

15. ABCD is a parallelogram, and O is any point outside it. Show that the *sum* or *difference* of the $\triangle^s$ OAB, OCD = $\frac{1}{2}$. par^{m} ABCD; and distinguish between the two cases.

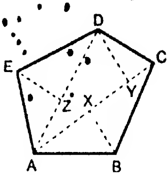
[Use the formula *area of* $\triangle = \frac{1}{2}ap$. Theorem [illegible]7. Cor. 1.]

16. (i) ABCD is a parallelogram, and O is any point *outside* the $\angle$BAD and its vertically opposite angle; prove that the $\triangle$OAC is equal to the sum of the $\triangle^s$ OAB, OAD.

(ii) If O is *within* the $\angle$BAD or its vertically opposite angle, prove that the $\triangle$OAC is equal to the difference of the $\triangle^s$ OAB, OAD.

II. Rectilineal Figures of more than Four Sides.

(i) A rectilineal figure may be divided into triangles whose areas can be separately calculated from suitable measurements. The sum of these areas will be the area of the given figure.

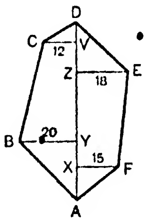


Example. The measurements required to find the area of the figure ABCDE are AC, AD, and the perp^s BX, DY, EZ.

(ii) The area of a rectilineal figure is also found by taking a **base-line** (AD in the diagram below) and offsets from it. These divide the figure into *right-angled* triangles and *right-angled* trapeziums, whose areas may be found after measuring the offsets and the various sections of the base-line.

Example. *Find the area of the inclosure* ABCDEF *from the plan and measurements tabulated below.*

	Yards	
	AD 56	
VC 12	AV 50	
	AZ 40	ZE 18
YB 20	AY 18	
	AX 10	XF 15



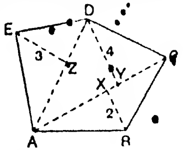
The measurements are made from A along the base line to the points from which the offsets spring.

		Sq. yds.	Sq. yds.
Here	△AXF = ½ AX × XF	= ½ × 10 × 15 =	75
	△AYB = ½ AY × YB	= ½ × 18 × 20 =	180
	△DZE = ½ . DZ × ZE	= ½ × 16 × 18 =	144
	△DVC = ½ . DV × VC	= ½ × 6 × 12 =	36
	trap^m XFEZ = ½ . XZ × (XF + ZE)	= ½ × 30 × 33 =	495
	trap^m YBCV = ½ . YV × (YB + VC)	= ½ × 32 × 32 =	512

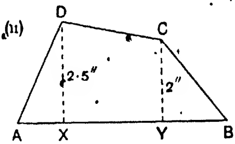
∴, by addition, the fig. ABCDEF = 1442 sq. yds.

EXERCISES.

1. Calculate the areas of the figures (i) and (ii) of which the plans and dimensions are given below.

(i) 

AC 6 cm., AD 5 cm.
Lengths of offsets figured in diagram

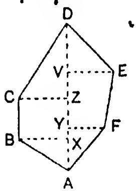
(ii) 

AX $=1\frac{1}{6}''$, XY $=2\frac{3}{4}''$
YB $=1\frac{3}{4}''$

2. Calculate the area of the inclosure ABCDEF from the plan and dimensions given below.

	YARDS	
	To D	
	40	
	30	12 to E
14 to C	20	
	10	9 to F
14 to B	6	
	From A	

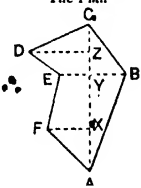
The Plan



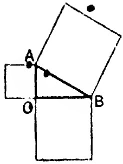
3. Find the area of the figure ABCDEF from the following measurements and draw a plan in which 1 cm. represents 20 metres.

	METRES.	
	To C	
	180	
80 to D	150	
40 to E	120	50 to B
60 to F	50	
	From A	

The Plan



EXERCISES LEADING TO THEOREM 29.



In the adjoining diagram, ABC is a triangle, *right-angled* at C, and squares are drawn on the three sides. Let us compare the area of the square on the hypotenuse AB with the *sum* of the squares on the sides AC, CB which contain the right angle.

1. Draw the above diagram, making AC = 3 cm., and BC = 4 cm.;

Then the area of the square on AC = 3^2, or 9 sq. cm.
and .... the square on BC = 4^2, or 16 sq. cm.

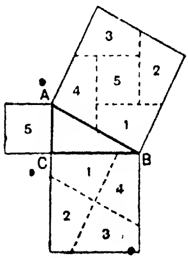
∴ the sum of the squares on AC, BC = 25 sq. cm.

Now *measure* AB; hence calculate the area of the square on AB, and compare the result with the *sum* already obtained.

2. Repeat the process of the last exercise, making AC = 1·0", and BC = 2·4".

3. If $a = 15$, $b = 8$, $c = 17$, show arithmetically that $c^2 = a^2 + b^2$.

Now draw on squared paper a triangle ABC, whose sides a, b, and c are 15, 8, and 17 units of length; and *measure the angle* ACB.



4. Take any triangle ABC, right-angled at C; and draw squares on AC, CB, and on the hypotenuse AB.

Through the mid-point of the square on CB (*i.e.* the intersection of the diagonals) draw lines parallel and perpendicular to the hypotenuse, thus dividing the square into four *congruent* quadrilaterals. These, together with the square on AC, will be found exactly to fit into the square on AB, in the way indicated by corresponding numbers.

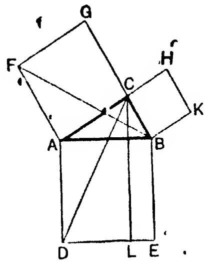
These experiments point to the conclusion that:

In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

A formal proof of this theorem is given on the next page.

THEOREM 29.

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.



Let ABC be a right-angled △, having the angle ACB a rt. ∠.

To prove that the square on the hypotenuse AB = *the sum of the squares on* AC, CB.

On AB describe the sq. ADEB, and on AC, CB describe the sqq. ACGF, CBKH

Through C draw CL par' to AD or BE.

Join CD, FB.

Proof. Because each of the ∠ˢ ACB, ACG is a rt. ∠,

∴ BC and CG are in the same st. line.

Now the rt. ∠ BAD = the rt. ∠ FAC;

add to each the ∠ CAB:

then the whole ∠ CAD = the whole ∠ FAB.

Then in the △ˢ CAD, FAB,

because { CA = FA,
AD = AB,
and the included ∠ CAD = the included ∠ FAB;

∴ the triangles are congruent, and equal in area.

Now the rect. AL is double of the △CAD, being on the same base AD, and between the same par^ls AD, CL.

And the sq. GA is double of the △FAB, being on the same base FA, and between the same par^ls FA, GB.

$\therefore$ the rect. AL = the sq. GA.

Similarly by joining CE, AK, it can be shown that

the rect. BL = the sq. HB.

$\therefore$ the whole sq. AE = the sum of the sqq. GA, HB :

that is, the square on the hypotenuse AB = the sum of the squares on the two sides AC, CB.

Q.E.D.

Obs. This is known as the Theorem of Pythagoras, a celebrated Greek mathematician of the sixth century B.C.

The result established may be stated as follows :

$$AB^2 = BC^2 + CA^2$$

That is, if the sides containing the right angle measure *a* and *b* units of length, and the hypotenuse *c* units,

$$c^2 = a^2 + b^2.$$

Hence $\quad a^2 = c^2 - b^2$; and $b^2 = c^2 - a^2$.

NOTE. The following important results should be noticed.

If CL and AB intersect in O, it has been shown in the course of the proof that

the sq. GA = the rect. AL,

that is, AC^2 = the rect. contained by AB, AO . . . . . (i)

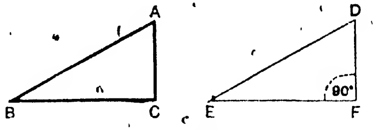
Also the sq. HB = the rect. BL,

that is, BC^2 = the rect. contained by BA, BO . . . . . . (ii)

[For Numerical Exercises on Theorem 29, see p. 141.
For Theoretical Exercises, see p. 142.]

THEOREM 30.

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these two sides is a right angle.



Let ABC be a triangle in which $AB^2 = BC^2 + CA^2$.

To prove that the $\angle C$ *is a right angle.*

Make EF equal to BC.

Draw FD perp. to EF, and make FD equal to CA.

Join ED.

Proof. Because $EF = BC$, and $FD = CA$,

$\therefore\ EF^2 = BC^2$, and $FD^2 = CA^2$.

Hence $EF^2 + FD^2 = BC^2 + CA^2$.

Now $EF^2 + FD^2 = DE^2$, since the $\angle F$ is a rt. $\angle$;

and $BC^2 + CA^2 = AB^2$, by hypothesis;

$\therefore\ DE^2 = AB^2$;

$\therefore\ DE = AB$.

Then in the $\triangle$s ACB, DFE,

because $\begin{cases} AC = DF, \\ CB = FE, \\ \text{and } AB = DE, \end{cases}$

$\therefore$ the $\triangle$s are congruent; so that the $\angle C =$ the $\angle F$.

But the $\angle F$ is a right angle, by construction;

$\therefore$ the $\angle C$ is a right angle.

Q.E.D.

NUMERICAL EXERCISES ON THEOREMS 29, 30.

1. Draw a triangle ABC, right-angled at C, having given:

(i) $a = 3$ cm., $b = 4$ cm.;
(ii) $a = 2{\cdot}5$ cm., $b = 6{\cdot}0$ cm.;
(iii) $a = 1{\cdot}2''$, $b = 3{\cdot}5''$.

In each case calculate the length of the hypotenuse c, and verify your result by measurement.

2. Draw a triangle ABC, right-angled at C, having given:

(i) $c = 3{\cdot}4''$, $a = 3{\cdot}0''$; [See Problem 8.]
(ii) $c = 5{\cdot}3$ cm., $b = 4{\cdot}5$ cm.

In each case calculate the remaining side, and verify your result by measurement.

3. Determine which of the following angles are right angled:

(i) $a = 14$ cm., $b = 48$ cm., $c = 50$ cm.;
(ii) $a = 40$ cm., $b = 10$ cm., $c = 41$ cm.;
(iii) $a = 20$ cm., $b = 99$ cm., $c = 101$ cm.

(Solve the following examples by calculation. In each case draw a plan, and verify the calculated result by measurement.)

4. A ladder whose foot is 9 feet from the front of a house reaches to a window-sill 40 feet above the ground. What is the length of the ladder?

5. A ship sails 33 miles due South, and then 56 miles due West. How far is it then from its starting-point?

6. Two ships are observed from a signal station to bear respectively N.E. 6·0 km. distant, and N.W. 1·1 km. distant. How far are they apart?

7. A ladder 65 feet long reaches to a point in the face of a house 63 feet above the ground. How far is the foot from the house?

8. B is due East of A, but at an unknown distance. C is due South of B, and distant 55 metres. AC is known to be 73 metres. Find AB.

9. A man travels 27 miles due South; then 24 miles due West; finally 20 miles due North. How far is he from his starting-point?

10. From A go West 25 metres, then North 60 metres, then East 80 metres, finally South 12 metres. How far are you then from A?

11. A ladder 50 feet long is placed so as to reach a window 48 feet high; and on turning the ladder over to the other side of the street, it reaches a point 14 feet high. Find the breadth of the street.

FURTHER EXERCISES ON THEOREMS 29, 30.

1. ABC is an isosceles triangle, right-angled at C; prove that

$$AB^2 = 2AC^2.$$

2. ABCD is a given square, and APQC is the square on the diagonal AC. Prove by Theorem 29 that the sq. APQC is double of the sq. ABCD.

Illustrate this from a figure, by showing that the sq. ABCD = twice the $\triangle$ ABC, and the sq. APQC = 4 times the $\triangle$ ABC.

3. If ABC is an equilateral triangle, and AX the perpendicular drawn from A to BC, show that

$$\text{(i) } AB = 2BX; \qquad \text{(ii) } AX^2 = 3BX^2.$$

4. How would you draw a square of which the area is equal to the sum of two given squares?

5. In the $\triangle$ ABC, AD is drawn perpendicular to the base BC. If the side c is greater than b,

$$\text{show that } c^2 - b^2 = BD^2 - DC^2.$$

6. If O is any point within a rectangle ABCD, prove that

$$OA^2 + OC^2 = OB^2 + OD^2.$$

[Through O draw XOY perp^r to AB or DC.]

7. ABC is a triangle right-angled at C, and the sides CA, CB are intersected by a straight line PQ. Join AQ, BP, and prove that

$$AQ^2 + BP^2 = AB^2 + PQ^2.$$

8. ABC is a triangle right-angled at C, and X, Y are the mid-points of CA, CB; show that

$$4(BX^2 + AY^2) = 5AB^2.$$

9. If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to BC, CA, AB respectively: show that

$$AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2.$$

10. In the figure of Theorem 29, join CE and prove that

$$CA^2 + CE^2 = CB^2 + CD^2.$$

11. In Theorem 29 let CL and AB intersect in O. If the figure (on a blackboard) is rubbed out, leaving only the rectangle ADLO, show how to re-construct the whole figure. [See Prob. 8, p. 110, observing that the hypotenuse subtends a right angle at a point on the semi-circle.]

12. Prove the following formula:

$$\text{diagonal of square} = \text{side} \times \sqrt{2}.$$

If the *side* = 5 cm., find the length of the *diagonal* in centimetres correct to three significant figures.

Why is it impossible to express the length of the diagonal *exactly* in terms of centimetres and decimals of a centimetre?

To what degree of accuracy can the diagonal be so expressed?

[Observe that though the side and diagonal of a square cannot be exactly expressed *arithmetically* in terms of a common unit, they can in theory be exactly drawn and compared *geometrically*.]

13. ABC is an equilateral triangle of which each side = $2m$ units, and the perpendicular from any vertex to the opposite side = p.

Prove that $p = m\sqrt{3}$.

When each side = 8 cm., find p correct to three significant figures.

What is meant by saying that p and m are *incommensurable*? [See p. 122.]

14. If in a triangle $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$, prove algebraically that $c^2 = a^2 + b^2$.

Hence by giving various numerical values to m and n, find sets of numbers representing the sides of right-angled triangles.

15. Work through the following verification of the Theorem of Pythagoras.

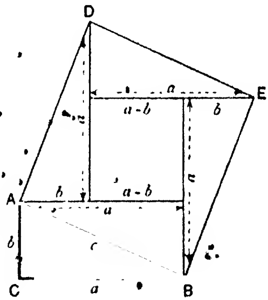
In the adjoining figure ABC is the given $\triangle$ rt.-angled at C, and ABED is the sq. on the hypotenuse AB.

By drawing par[ls] to BC, CA, as in the fig., it will be seen that the sq. ABED is divided into *four* rt.-angled $\triangle$s together with a central square.

Prove that these four $\triangle$s are congruent, and express the area of each in terms of a and b.

Express the central square in terms of a and b. Hence show that

$$c^2 = a^2 + b^2.$$



[The method given below may be omitted from a first course.]

AREA OF A TRIANGLE.

Given the lengths of the three sides of a triangle, to calculate the AREA *by means of Theorem 29.*

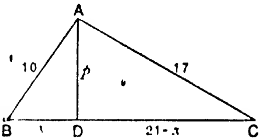
EXAMPLE. *Find the area of a triangle whose sides measure* 21 m., 17 m., *and* 10 m.

Let ABC represent the given triangle.

Draw AD perp. to BC, and denote AD by p.

We shall first find the length of BD.

Let BD $= x$ metres; then DC $= 21 - x$ metres.



From the right-angled △ ADB, we have by Theorem 29

$$AD^2 = AB^2 - BD^2 = 10^2 - x^2.$$

And from the right-angled △ ADC,

$$AD^2 = AC^2 - DC^2 = 17^2 - (21 - x)^2,$$

$$\therefore\ 10^2 - x^2 = 17^2 - (21 - x)^2$$

$$100 - x^2 = 289 - 441 + 42x - x^2,$$

whence $\qquad x = 6.$

Again, $\qquad AD^2 = AB^2 - BD^2,$

or $\qquad p^2 = 10^2 - 6^2 = 64;$

$$\therefore\ p = 8.$$

Now $\qquad$ *area of triangle* $= \frac{1}{2} \cdot$ *base* $\times$ *altitude*

$$= (\tfrac{1}{2} \times 21 \times 8) \text{ sq. m.} = 84 \text{ sq. m.}$$

EXERCISES.

1. In a triangle ABC, AD is drawn perpendicular to BC. Let p denote the length of AD.

(i) If $a = 25$ cm., $p = 12$ cm., BD $= 9$ cm.; find b and c.

(ii) If $b = 41''$, $c = 50''$, BD $= 30''$; find p and a.

And prove that $\sqrt{b^2 - p^2} + \sqrt{c^2 - p^2} = a$.

2. In the triangle ABC, AD is drawn perpendicular to BC. Prove that

$$c^2 - BD^2 = b^2 - CD^2.$$

If $a = 51$ cm., $b = 20$ cm., $c = 37$ cm.; find BD.

Thence find p, the length of AD, and the area of the triangle ABC.

Find by the method of the last Exercise the area of the triangles whose sides are as follows:

3. 20 ft., 13 ft., 11 ft.
4. 15 yds., 14 yds., 13 yds.
5. 21 m., 20 m., 13 m.
6. 30 cm., 25 cm., 11 cm.
7. 37 ft., 30 ft., 13 ft.
8. 51 m., 37 m., 20 m.

9. A straight rod PQ slides between two straight rulers OX, OY placed at right angles to one another. In one position of the rod OP = 5·6 cm., and OQ = 3·3 cm. If in another position OP = 4·0 cm., find OQ graphically; and test the accuracy of your drawing by calculation.

10. ABC is a triangle right-angled at C, and p is the length of the perpendicular from C on AB. By expressing the area of the triangle in two ways, show that

$$pc = ab.$$

Hence deduce
$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

11. Find by the method of the preceding page the areas of the triangles whose sides are as follows:

(i) $a = 17''$, $b = 10''$, $c = 9''$. (ii) $a = 25$ ft., $b = 17$ ft., $c = 12$ ft.
(iii) $a = 41$ cm., $b = 28$ cm., $c = 15$ cm. (iv) $a = 40$ yd., $b = 37$ yd., $c = 13$ yd.

12. In the figure of the preceding page, if the given sides are a, b and c units in length, prove

(i) $x = \dfrac{a^2 + c^2 - b^2}{2a}$, (ii) $p^2 = c^2 - \left(\dfrac{a^2 + c^2 - b^2}{2a}\right)^2$.

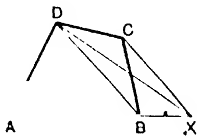
(iii) $\triangle = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$.

SECTION II. CONSTRUCTIONS.

ON AREAS.

PROBLEM 10

To draw a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral.

To construct a triangle equal in area to ABCD *in area.*

Construction. Join DB.

Through C draw CX par[l] to DB, meeting AB produced in X

Join DX

Then DAX is the required triangle.

Proof. Now the △s XDB, CDB are on the same base DB and between the same par[ls] DB, CX :

∴ the △ XDB = the △ CDB in area.

To each of these equals add the △ DAB ;

then the △ DAX = the fig. ABCD.

EXERCISES.

(N.B. Give the construction in each case. A full formal proof is only required when specially asked for; but in every example the reason for the construction should be briefly explained.)

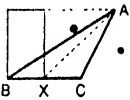
1. Draw a square on a side of 5 cm., and make a parallelogram of equal area on the same base, and having an angle of 45°.
Find (i) by calculation, (ii) by measurement the length of an oblique side of the parallelogram.

2. Draw any parallelogram ABCD in which AB = $2\frac{1}{2}''$ and AD = 2", and on the base AB draw a rhombus of equal area.

3. ABC is a given triangle. Show how to draw on the base BC (i) a right-angled triangle; (ii) an isosceles triangle, each equal in area to the △ ABC.

4. Through A, the vertex of a △ ABC, draw a straight line dividing the given triangle into two triangles of equal area. When will such triangles be congruent?

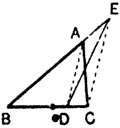
5. Prove that a triangle and a parallelogram are equal in area if they have the same altitude, and if the base of the triangle is double that of the parallelogram.
Hence show how to draw an isosceles triangle equal in area to a given rectangle.



6. From the adjoining figure devise a construction for drawing a rectangle equal in area to the △ ABC. Give a formal proof.

7. Construct a triangle ABC in which a = 7.5 cm., b = 7.0 cm., c = 6.5 cm.; and draw a rectangle of equal area.

8. Draw an equilateral triangle on a side of 2.5", and construct a parallelogram of equal area and having one angle equal to 120°.



9. From the adjoining figure devise a construction for drawing a triangle equal in area to a given △ ABC, and having its base BD of given length. (D lies in BC, or BC produced.)

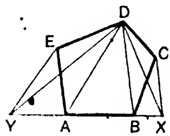
10. Construct a triangle equal in area to a quadrilateral ABCD, having its vertex at a given point X in DC, and its base in the same straight line as AB.
[Through C and D draw CE, DF parᵗ to BX and AX respectively, and meeting AB produced in E and F. Prove that EXF is the triangle required.]

Reduction of a Rectilineal Figure to a Triangle or Rectangle of Equal Area.

By the method of Problem 10 it is always possible to draw a polygon equal in area to a given polygon, and having fewer sides by one than the given figure; and thus, step by step, any polygon may be reduced to a triangle of equal area.

For example, in the adjoining diagram the five-sided fig. EDCBA is equal in area to the four-sided fig. EDXA.

The fig. EDXA may now be reduced to the equivalent DXY.



Exercises

1. Draw a quadrilateral ABCD from the following data:

AB = BC = 5·5 cm., CD = DA = 4·5 cm., the ∠A = 75°.

Reduce the quadrilateral to a triangle of equal area. Measure the base and altitude of the triangle, and hence calculate the approximate area of the given figure.

2. Draw a quadrilateral ABCD having given:

AB = 2·8″, BC = 3·2″, CD = 3·3″, DA = 3·6″, and the diagonal BD = 3·0″.

Construct an equivalent triangle, and hence find the approximate area of the quadrilateral.

3. On a base AB, 4 cm. in length, describe an equilateral pentagon (5 sides), having each of the angles at A and B 108°.

Reduce the figure to a triangle of equal area, and by measuring its base and altitude, calculate the approximate area of the pentagon.

4. A quadrilateral field ABCD has the following measurements:

AB = 450 metres, BC = 380 m., CD = 330 m., AD = 390 m.,

and the diagonal AC = 660 m.

Draw a plan (scale 1 cm. to 50 metres). Reduce your plan to an equivalent triangle, and measure its base and altitude. Hence estimate the area of the field.

5. By what steps would you reduce a rectilineal figure of four or more sides to a rectangle of equal area?

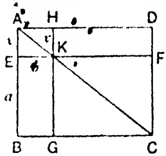
For example, draw a six-sided figure (hexagon) having each side 2 inches in length, and each angle 120°; and reduce the hexagon to an equivalent triangle.

Hence construct a rectangle of equal area.

Deduce the approximate area of the hexagon, after making any necessary measurements.

On the Construction of a Rectangle of Given Area and having one Side of Given Length.

Preliminary Theorem. In AC, a diagonal of the rectangle ABCD, take any point K; and through K draw EF, HG par' to AD and AB, as in the adjoining figure.



Now observe that the diagonal AKC bisects each of the rects. EH, GF, BD, and hence prove that the rect. EG = the rect. HF in area.

This property, which is common to all parallelograms [see Ex. 1, p. 130], enables us **to construct a rectangle equal in area to a given rectangle and having one side of given length.**

For example. Let EBGK be the given rectangle (see figure above) whose sides EB, EK are a and b units in length. Required to construct a rectangle of equal area and having one side x units in length.

Produce BE to A, making EA = x units in length. Join AK. Produce AK and BG to meet at C. Through A draw a par' to BC; and through C draw a par' to BA. let these par's meet at D. Produce EK to meet DC at F, and produce GK to meet AD at H.

Then, as above, the rect. HF = the rect. EG in area, and one of its sides HK = x units of length.

EXERCISES.

1. Draw a rectangle ABCD having AB = 8 cm. and AD = 3 cm.; and construct a rectangle of equal area, having one side 6 cm. in length.

2. Given a parallelogram ABCD, in which AB = 2·4", AD = 1·8", and the $\angle$A = 55°. Construct a parallelogram equiangular to ABCD and of equal area, the greater side measuring 2·7". Measure the shorter side.

Repeat the process giving to A any other value, and compare your results. What conclusion do you draw?

3. Draw a rectangle on a side of 5 cm. equal in area to an equilateral triangle on a side of 6 cm. Measure the remaining side of the rectangle, and calculate its approximate area.

4. Draw a quadrilateral ABCD from the following data: $\angle$A = 90°, AB = 7·7 cm., AD = 3·6 cm., BC = 6·8 cm., DC = 5·1 cm.

Construct a triangle of equal area, and hence draw a rectangle equal in area to the quadrilateral ABCD, and having one side 4 cm. in length.

MISCELLANEOUS CONSTRUCTIONS.

1. Draw a square on a diagonal of 6 cm. Show that each of its sides is $3\sqrt{2}$ cm. in length, and that its area is 18 sq. cm.

2. Draw an equilateral triangle on a base of 8 cm.; and construct a parallelogram of the same altitude and area, and having one angle 120°.

Show that your construction divides the triangle into two parts which may be fitted together over the parallelogram.

3. How would you draw a square equal in area to the sum of two squares?

If the *areas* of the two squares are 36 sq. cm. and 64 sq. cm., construct a square of area equal to their sum. Measure and calculate its side.

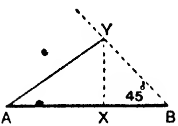
4. Express the number 13 as the sum of two square numbers; hence show how to draw a square containing 13 square inches. Point out in your figure a line represented by $\sqrt{13}$ inches.

5. By the method of the last Exercise draw lines that may be represented by (i) $\sqrt{10}$, (ii) $\sqrt{29}$, (iii) $\sqrt{34}$ units.

6. By means of Problem 8, p. 110, show how to construct a square equal in area to the difference between two given squares.

The *sides* of two squares are 2·5 in. and 0·7 in., construct a square equal in area to the difference of the two squares. Verify your drawing by measurement and calculation.

7. From the figure in the margin show how to divide a given straight line AB into two parts such that the sum of their squares may be equal to a given square.

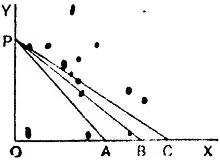


[Suppose AY = side of given square, and X is the required point of division. Note that XY must = XB, so that $\angle$XBY must = 45°. Hence devise the required construction.]

For example: Divide a line 4·6" in length into two parts so that the sum of their squares may be equal to the square on a side of 3·4".

8. Show how to divide a straight line AB at X, so that $AX^2 = 2XB^2$.

9. Show how to draw straight lines whose lengths may be represented by $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots$



[Take OX, OY at right angles to one another, and from them mark off OA, OP, each *one* unit of length. Join PA, and prove PA = $\sqrt{2}$.

From OX mark off OB equal to PA. Join PB, and prove PB = $\sqrt{3}$.

From OX mark off OC equal to PB. Join PC, and prove PC = $\sqrt{4}$. By continuing the process lines represented by $\sqrt{5}, \sqrt{6}, \sqrt{7}, \ldots$ may be drawn.]

10. Draw a △ABC in which $a = 8$ cm., $b = 6$ cm., $c = 7$ cm.; and divide it into *five* triangles of equal area by straight lines drawn through A.

11. From a given △ABC cut off *one-half* by a straight line drawn through P, a point in the side AB.

[Bisect AB at Z. Join CZ, CP. Through Z draw ZQ par[l] to PC. Join PQ, and show that △PBQ = $\frac{1}{2}$ △ABC.]

12. Show how to trisect a given △ABC by lines drawn from X, a point in the side BC.

[Trisect BC at the points P, Q by Prob. 6. Join AX, and through P and Q draw PH and QR par[l] to AX. Join XH, XK. These lines trisect the triangle, as may be shown by joining AP, AQ.]

13. Cut off from a given triangle a fourth, fifth, sixth, or any part required by a straight line drawn from a given point in one of its sides.

14. **Bisect a quadrilateral by a straight line drawn through an angular point.**

[Reduce the quadrilateral to a triangle of equal area, and join the vertex to the middle point of the base.]

15. Cut off from a given quadrilateral a third, a fourth, a fifth, or any part required, by a straight line drawn through a given angular point.

MISCELLANEOUS EXAMPLES ON SECTION II.

1. In a quadrilateral ABCD, the $\angle A = 90°$, and the diagonal BD is perpendicular to one of the sides. Show that the square on the longest side of the quadrilateral is equal to the sum of the squares on the other three sides.

2. The sides BC, CA, AB of a triangle are produced to D, E, F, so that CD = BC, AE = CA, BF = AB. Prove that the area of $\triangle$ DEF is seven times the area of $\triangle$ ABC.

3. The side AB of a parallelogram ABCD is produced to E. Prove that

$$\triangle CED + \triangle BED = \triangle AEC.$$

4. In a quadrilateral ABCD it is given that the diagonals are at right angles, and also that $AB^2 + BC^2 = AD^2 + DC^2$. Prove that AC bisects BD.

5. The sides AB, AC of a triangle are at right angles, and squares ABDE, ACFG are described outwards on them; if DE, FG are produced to meet in X, prove that XA is perpendicular to BC and equal to it.

6. ABCD, ABEF are parallelograms with a common side AB. If CE, DF are joined, show that CDFE is a parallelogram, and that its area is either the sum or the difference of the areas of the given parallelograms.

7. ABC is a triangle in which BA, CA are produced to D and E, making AD equal to AB and AE equal to AC. If P is any point in BC, show that the sum of the triangles PAD and PAE is constant.

8. An equilateral triangle ABD is described on the side AB of a triangle ABC, right angled at B (D falling outside $\triangle$ ABC); prove that the area of $\triangle$ DBC = $\frac{1}{2}$ area of $\triangle$ ABC.

9. ABCD is a square, M and R are the mid-points of BC and CD respectively. Show that $\triangle$ AMR is three-eighths of the square in area.

10. A haystack, of which the edge of the thatch is 15 ft. vertically above the ground, overhangs its base by 3 ft. measured horizontally. What length of ladder will be required so that it will just reach to the edge of the thatch, if placed at a point on the ground 11 ft. from the base of the haystack?

11. ABC is a triangle having AB = AC, P is any point on BC; PX and PY are perpendiculars on the sides AB and AC. By expressing the area of the triangle in two ways, show that the sum of PX and PY is the same for all positions of P.

12. Construct a parallelogram with two adjacent sides 3 in. and 2 in. long, the distance between the longer sides being 1·5 in. Find the distance between the other two parallels.

13. ABCD is a square on a side of m inches. On the sides AB, AD, CB, CD respectively points E, H, F, G are taken so that

$$AE = AH = CF = CG = \tfrac{1}{3}m.$$

Show that EFGH is a rectangle the area of which is $\frac{4}{9}m^2$ square inches.

14. If two unequal right-angled triangles ABC, ADC are drawn on opposite sides of their common hypotenuse AC, and if AM, CN are perpendiculars to BD, cutting it in M, N, prove that

$$BM^2 + BN^2 = DM^2 + DN^2.$$

15. A parallelogram has sides 2″ and 3″ long at an angle of 45°. Construct a rhombus of equal area. Find the area correct to the nearest tenth of a square inch.

16. Construct a triangle ABC whose sides are 4·5 cm., 3·4 cm., and 6·3 cm., and find a point P within the triangle such that the triangles PBC, PCA, PAB are equal in area.

17. ABCD, AEFG are two parallelograms having a common point at A, and having the vertex E on BC, and the vertex D on FG. Prove that the parallelograms are equal in area. [Produce DC, GF to meet.]

18. AD, BE, CF are medians of a △ABC, right-angled at B, and they meet at G. Prove that

(i) $\triangle AED = \frac{1}{8}$ rect. AB, BC. (ii) $AD^2 = EB^2 + 3ED^2$.

19. ABC is a triangle in which $\angle B = 45°$, $\angle C = 60°$, and AD is the perpendicular from A on BC. Prove that

$$2AB^2 = 3AC^2 = 4AD^2.$$

20. Construct a trapezium whose sides are 10, 4, 7, 6 cm. in length, having the first and third sides parallel.

Measure the height of the trapezium and find its area.

21. Construct a parallelogram on a base of 3″, with diagonals 5″ and 3·2″. Make a triangle of the same area on the same base, having an angle of 45°.

22. Construct a quadrilateral ABCD having AB = 2·6 cm., AC = 5 cm., $\angle ABC = 108°$, $\angle BAD = 130°$, $\angle CDA = 90°$.

23. Construct a triangle equal in area to the quadrilateral ABCD, having AB as one side, and the opposite vertex on BC produced.

24. Construct a quadrilateral ABCD from the following data:

AB = 2″, AD = 3″, $\angle BAD = 90°$, the diagonal AC = $3\frac{3}{4}$″, and the area of the quadrilateral is 6 square inches. State the construction without proof.

25. The medians drawn from the points B, C of the triangle ABC are at right angles; show that $CA^2 + AB^2 = 5CB^2$.

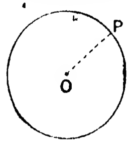
SECTION III.

Loci.

Definition. The locus of a point is the path traced out by it when it moves in accordance with some given condition.

Example 1. Suppose the point P to move so that its distance from a fixed point O is constant (say 1 centimetre).

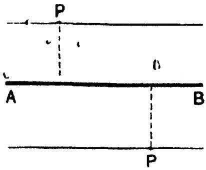
Then the locus of P is evidently the circumference of a circle whose centre is O and radius 1 cm.



Notice (i) that every point whose distance from O is 1 cm. is on the circumference; (ii) that every point on the circumference is 1 cm. distant from O.

Example 2. Suppose the point P moves at a constant distance (say 1 cm.) from a fixed straight line AB.

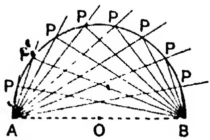
Then the locus of P is one or other of two straight lines parallel to AB, on either side, and at a distance of 1 cm. from it.



When we find a series of points which satisfy the given condition, and through which therefore the moving point must pass, we are said to plot the locus of the point. Having plotted a sufficient number of such points, we may connect them by a continuous line drawn free-hand. This line represents the required locus.

Example 3. A and B are two fixed points. P is a point that moves so that in all positions the $\angle APB$ is a right angle. Plot the locus of P.

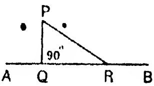
Through A draw a series of straight lines, say at intervals of 10°. From B draw a perpendicular BP to each of the lines through A. We thus get a series of points P, all of which lie on the required locus. This locus will be found to be the circumference of a circle on the diameter AB.



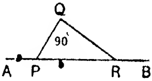
EXERCISES.

[*N.B. Squared (inch) paper may be used with advantage for many of the following exercises, especially for those in which a series of perpendiculars are to be drawn to a fixed horizontal or vertical line.*]

1. Along a straight ruler, marked AB in the figure, slide a set square so that one side containing the right angle keeps against the straight-edge. Watch the opposite vertex P. What locus does it trace out, and why?



2. Slide the set square along the ruler so that the hypotenuse keeps in contact with the straight-edge. Again watch the vertex Q, and say what locus it traces out, and why.



3. How many points could be plotted at a distance of 5 cm. from a fixed point O? On what line would all these points lie? Draw the line.

4. From a given position O a point P moves along a straight line at a uniform rate of 1·5 cm. per second, but in an *unknown direction.* On what line must P be found at the end of 3 seconds?

5. On squared paper rule a vertical straight line AB, and (i) draw the locus of a point that moves on the right of AB, keeping at a distance of 6 cm. from it. (ii) Draw the locus of a point that moves on the left of AB, keeping at a distance of 3 cm. from it.

What is the relation of these loci to AB, and to one another?

6. On squared paper rule a vertical straight line AB, and plot a number of points each distant 2 in. from it, there being no restriction as to the side of AB on which they lie. What line or lines would pass through all such points?

7. Draw on squared paper two horizontal parallels AB, CD of indefinite length and 3 inches apart. Mark any point P whose perpendicular distance from AB is 1·5". If P moves so as always to be equidistant from AB and CD, what locus does it trace out?

8. AB is a fixed straight line of indefinite length. QP is a straight line 8 cm. long, and one end Q moves along AB. Draw the locus of P when

(i) QP keeps perpendicular to AB throughout the movement;

(ii) QP is inclined to AB at a constant angle of 60°.

9. LM is a straight bar 3 feet long, and O is a fixed pivot in it 2 feet from L. If the bar rotates about O, what is the locus of L? What is the locus of M?

Draw a figure (1 inch to 1 foot).

If the bar were bent at O so that the parts LO, OM made a constant angle, say of 135°, what would be the loci of L and M, as the bent bar revolved about O?

Illustrate by a figure.

10. A bar OQP is bent so that the parts OQ, QP make a constant angle, say of 120°, at Q. The bent bar revolves about O. What is the locus of P? Illustrate by a sketch, and give a reason for your answer.

11. Draw (on squared paper) a triangle ABC, making BC horizontal, and $a = 2.5''$, $b = 3.0''$, $c = 4.5''$. Rule at about equal intervals some 5 or 6 straight lines parallel to BC and terminated by AB and AC, or these lines produced. Call each parallel LM, and mark its middle point P. Draw a line passing through all the points marked P.

12. With a fixed point O as centre and with radius 2 inches draw a circle and let AB be a horizontal diameter. Rule chords parallel to AB, 4 or 5 on each side; and in each chord mark the middle point P. What line will pass through all the points marked P?

13. Draw a rectangle OQPR, making OQ = 10 cm., and OR = 8 cm. Suppose the rectangle to revolve about a pivot at O. What is the locus of Q, of R, of P? What is the length of OP?

14. On squared paper take a fixed horizontal straight line AB of indefinite length, and let X be a point on it moving from A towards B. Suppose XP is perpendicular to AB and that in all of its positions XP = XA. Plot the locus of P.

[Do this by drawing perpendiculars to AB at points (X) distant 0.5", 1.0", 1.5", 2.0", etc., from A; and make XP 0.5", 1.0", 1.5", etc., in length in the corresponding positions. Then draw a line through all the positions of P.]

15. As in the last Exercise, AB is a fixed straight line, and X a point on it moving from A towards B. XP is a line of variable length perpendicular to AB. Plot the locus of P when in all positions

(i) XP = 2XA; (ii) XP = 3XA.

16. Draw a straight line AB of indefinite length, and mark a fixed point O 3 inches from it. Let Q be *any* point in AB, and P the middle point of OQ. If Q moves along AB, plot the locus of P.

[Do this by drawing OQ in a series of positions (say 5 or 6) on each side of the perpendicular from O to AB; and in each position of OQ mark its middle point P. Draw a line through which all these middle points lie.]

17. Draw two parallel straight lines AB, CD, 5 cm. apart; and draw a straight line LM, of fixed length 6 cm., having its ends L and M on AB and CD respectively. If LM slides along the parallels, plot the locus of its middle point P.

18. Mark a fixed point A, and draw several circles of radius 4 cm., each passing through A. What line would pass through the centres of *all* the circles that could be so drawn?

19. Take two fixed points A and B 6 cm apart, and mark X the mid-point between them. Then find two points each distant 3·5 cm from both A and B (by drawing intersecting arcs from A and B as centres and with radius 3·5 cm) Similarly plot *pairs* of points whose distances from A and B are 4·0 cm., 4·5 cm, 5·0 cm, 5·5 cm, and so on.

Note that each of these points is *equidistant* from A and B Draw a line through them and say how this line is related to the straight line AB.

20. An isosceles triangle PBC stands on a fixed base BC of length 5 cm. If the altitude of the triangle varies, what is the locus of P?

21. Draw two straight lines AX, AY of indefinite length, making a fixed angle at A, say of 60°. Plot a series of 6 points at distances from both AX and AY of 1 cm, 2 cm, 3 cm, etc This may be done for the present purpose by trial Note that each such point of intersection is *equidistant* from AX and AY. Draw a line passing through them all, and observe its relation to AX and AY

22. Repeat the last Exercise, making the $\angle XAY$ 120°.

23. How many parallelograms could be drawn on a fixed base 2 inches long, and having an altitude of 1·5 inches? Draw six such parallelograms, and in each one mark the point of intersection of its diagonals. Draw a line passing through all these points.

24. Draw a circle of radius 5 cm, and take a fixed point O on its circumference. Suppose Q to be a moving point on the circumference. Plot the locus of P, the mid-point of OQ.

We may now widen our conception of a locus by regarding it not merely as the path of a point moving under a given condition, but as an *aggregate of points* comprising all that satisfy the given condition and none that do not satisfy it

Therefore, in order to show that the locus of points satisfying some given condition is a certain line (or lines), we must in strictness prove

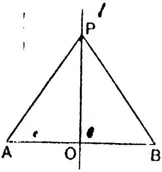
(i) that every point satisfying the given condition lies on that line;

(ii) that every point on that line satisfies the given condition.

NOTE. In this extended view of a locus, the locus of P in Example 2, p. 154, consists of *both* the parallels to AB, and not merely *one or other* of them.

THEOREM 31.

The locus of points equidistant from two fixed points A *and* B *is the straight line bisecting* AB *at right angles.*



One point on the locus will clearly be O, the middle point of AB. Observe that O is a *fixed* point

(i) Let P be *any* other point such that PA = PB.

Join OP

To prove that OP *is perpendicular to* AB

Outline of Proof. Show that the $\triangle$s POA, POB are congruent by Theorem 13,

so that $\angle$POA = $\angle$POB;

hence PO is perpendicular to AB.

That is, every *point* P *which is equidistant from* A *and* B *lies on the straight line bisecting* AB *at right angles.*

(ii) Let P be *any* point in OP the perpendicular bisector of AB.

Join PA, PB.

To prove that P *is equidistant from* A *and* B.

Show that the $\triangle$s POA, POB are congruent by Theorem 9;

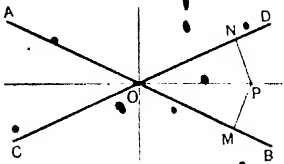
so that PA = PB.

That is, every *point on the perpendicular through* O *is equidistant from* A *and* B.

$\therefore$ *the perpendicular bisector of* AB *is the required locus.*

THEOREM 32.

The locus of points equidistant from two given straight lines AB, CD, *intersecting in* O, *is the pair of lines which bisect the angles between* AOB, COD.



Let P be *any* point equidistant from AB and CD; that is, let the perp PM = the perp PN.

(i) *To prove that* P *is on the bisector of the* ∠BOD (*or* ∠AOD).

Join P to O, the intersection of AB, CD.

Then in the △s PMO, PNO,

because { the ∠s PMO, PNO are *right angles*,
the hypotenuse OP is common,
and one side PM = one side PN;

∴ the triangles are congruent; *Theor.* 18.

so that the ∠POM = the ∠PON.

That is, P lies on the bisector of the ∠BOD.

(ii) Let P be *any* point on the bisector of the ∠BOD (or ∠AOD).

To prove that P *is equidistant from* AB, CD.

Show that the △s PMO, PNO are congruent by Theorem 10;

∴ so that PM = PN.

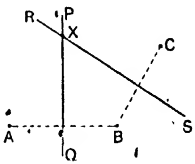
That is, P is equidistant from AB, CD.

It follows that *the required locus is the pair of lines which bisect the angles between* AB *and* CD.

Intersection of Loci.

The method of Loci may be used to find the position of a point which is subject to two conditions. For corresponding to each condition there will be a locus on which the required point must lie. Hence all points which are common to these two loci, that is, all the points of intersection of the loci, will satisfy *both* the given conditions.

Example. *To find a point equidistant from three given points* A, B, C, *which are not in the same straight line.*



(i) The locus of points equidistant from A and B is the straight line PQ, which bisects AB at right angles.

(ii) Similarly, the locus of points equidistant from B and C is the straight line RS which bisects BC at right angles.

Hence the point common to PQ and RS must satisfy both conditions: that is to say, X the point of intersection of PQ and RS will be equidistant from A, B, and C.

EXERCISES.

(*On Theorems* 31, 32.)

1. Take two fixed points A and B 3 inches apart. Draw the locus of points equidistant from A and B. Find a point on the locus 2·5 inches from A and B.

2. If a number of isosceles triangles stand on the same base BC, what line will pass through the vertices of them all?

Take BC = 2·6", and draw isosceles triangles whose altitudes are respectively 2·0", 2·5", and 3·0".

3. A point P moves along a straight line RQ; find the position in which it is equidistant from two given points A and B. What difficulty arises if RQ is perpendicular to AB?

4. A and B are two fixed points within a circle: find points on the circumference equidistant from A and B. How many such points are there?

5. Take two straight lines OA, OB, making the $\angle O = 72°$; and draw the locus of points equidistant from them. Find a point P distant 4 cm. from both OA and OB.

6. Two straight lines AOB, COD intersect at O, making the $\angle BOD = 60°$. Draw the locus of points equidistant from them. How many points can be found distant 1·5″ from both? Illustrate your answer by a figure.

7. A point P moves along a straight line RQ; find the position in which it is equidistant from the intersecting straight lines OA, OB. Is this always possible?

8. Draw a circle of radius 1·5 inches, and suppose that a point P moves so that its distance from the circumference (measured radially) is always 0·5 inch. Show by a figure that the locus of P consists of *two* circles. How are they situated?

If the distance of P from the circumference were 1·5″, what would the locus then be?

9. A and B are two fixed points 6 cm. apart. Find points which are 4 cm. distant from A, and 5 cm. from B. Explain the construction by the method of loci.

10. AB and CD are two straight lines intersecting at O. Find points 3 cm. distant from AB, and 4 cm. from CD. How many solutions are there?

11. OQ and OR are two straight lines, of fixed and equal lengths, revolving about O at the same rate in opposite ways. If they start together from a fixed direction OA, what is the locus (i) of R, (ii) of Q, (iii) of P, the middle point of RQ?

12. Draw a triangle ABC from the following data: $a = 3·5″$, $b = 4·0″$, $c = 2·5″$. Draw the locus of points equidistant from B and C; also the locus of points equidistant from C and A. If these loci intersect at S, show that SA = SB = SC.

13. Draw a triangle ABC of any shape, and by the method of the last Exercise, find a point equidistant from A, B, and C.

14. Take two fixed points A and B 2·0″ apart, and plot points P, Q, R distant respectively 1·5″, 2·0″, 2·5″ from both A and B. From centres P, Q, R with 1·5″, 2·5″, 2·0″ as radii draw circles. Why will these circles pass through A and B? What is the locus of the centres of *all* circles which pass through A and B?

15. Given a triangle ABC of any shape. Draw the locus of points equidistant from AB, AC; and hence find a point in BC equidistant from the other sides.

16. Draw a $\triangle$ ABC from the following data: $a = 8·5$ cm., $b = 10·0$ cm., $c = 7·0$ cm. Draw the locus of points equidistant from BA, BC; also the locus of points equidistant from CA, CB.

If these loci cut at I, show that I is equidistant from the three sides of the triangle.

H. Sh. G.

17. Draw a triangle of any shape, and find a point I which is equidistant from the three sides.

18. Draw a triangle ABC of any shape; and find a point P equidistant from B and C, and also equidistant from AB, AC.

19. Draw two parallel straight lines AB, CD 3 inches apart, and take a point O between them 0·5 inch from AB. Find (if possible) a point or points equidistant from AB and CD, and at a distance from O of

(i) 2 inches, (ii) 1 inch, (iii) 0·5 inch.

How many solutions will there be in each case? Use the same figure.

20. On a given base construct a triangle of given altitude, having its vertex on a given straight line.

(The Examples given below are to be worked theoretically; that is, without plotting, unless otherwise stated. It will be sufficient to prove that every point satisfying the given condition lies on the supposed locus, as in these instances the converse follows readily.)

21. A is a fixed point, and a variable point Q moves on a fixed straight line BC. Prove that the locus of P, the middle point of AQ, is a straight line parallel to BC.

[Draw AX perpendicular to BC, and bisect AX at R. Note that R is a fixed point. Join RP. Then see Theorem 23.]

22. A is a fixed point, and Q moves on a fixed straight line BC. Join AQ and produce it to P, making QP = AQ. Find the locus of P, and give a theoretical proof.

23. AB and CD are parallel straight lines, and O is a fixed point. Through O a series of straight lines are drawn to cut the parallels at X and Y. Find the locus of P, the middle point of XY, (i) when O is outside the parallels; (ii) when O is between them.

24. From a fixed point O as centre and with radius 6 cm. draw a circle, and mark a fixed point A 4 cm. from O. If a variable point Q moves round the circumference of the circle, prove that the locus of P, the middle point of AQ, is a circle.

[Join AO, QO, and bisect AO at R. Note that R is a fixed point, and OQ of fixed length.]

25. If in Exercise 24 the point O were on the circumference, or outside it, prove that the locus would still be a circle.

26. Draw a circle and take a fixed point A on its circumference. Let Q be a point moving round the circumference. Join AQ and produce it to P, making QP = AQ. Find the locus of P, and give a theoretical proof.

27. A straight rod of given length (say 3·4") slides between two straight rulers placed at right angles to one another.

Plot the locus of its middle point; and *prove* that this locus is the fourth part of the circumference of a circle.

28. PBC is a triangle on a fixed base, BC, which is produced to D. If the vertex P moves so that the $\angle$P is always one-half of the exterior $\angle$PCD, prove that its locus is a circle.

29. ABCD is a parallelogram made of rods connected by pivots. If AB is fixed, find the locus of the middle point of CD.

30. OX, OY are straight lines of indefinite length at right angles to one another. RS, a line of fixed length, slides between OX and OY. RP and SP are perpendicular to OX and OY. Prove that the locus of P is a circle.

31. ABC is a given triangle, and the point Q moves on BC. From Q (in any one of its positions) QR, QS are drawn parallel to CA, BA. Find the locus of P, the middle point of RS, and give a theoretical proof.

32. ABCP is a quadrilateral, of which the lengths of the sides AB, BC, CP are 8·0 cm., 7·0 cm., 4·0 cm., and the diagonal AC = 6·5 cm. Prove that the locus of P is a circle. What are its centre and radius?

33. CAB is one of a series of isosceles triangles on a fixed base AB. If AC is produced to P making CP = AC, find the locus of P for varying positions of C.

34. Two straight lines OX, OY cut at right angles, and from P, a point within the angle XOY, perpendiculars PM, PN are drawn to OX, OY respectively. Plot the locus of P when

(i) PM + PN is constant (= 6 cm., say),
(ii) PM − PN is constant (= 3 cm., say).

In each case give a theoretical proof of the result you arrive at experimentally.

35. Two straight lines OX, OY cut at right angles; and Q and R are points in OX and OY respectively. Plot the locus of the middle point of QR, when

(i) OQ + OR = constant (= 6 cm., say);
(ii) OQ − OR = constant (= 3 cm., say).

Give a theoretical proof of your result.

36. The $\triangle$PBC stands on a fixed base BC. What is the locus of the vertex P, if in all positions the area of the triangle is constant?

For example: If the constant area = 22·5 sq. cm., and BC = 7·5 cm., calculate the altitude and draw the locus of P.

If, in addition, BP = 10·0 cm., construct the triangle.

37. Draw a triangle ABC, having given $a = 3·0''$, $b = 2·5''$, $c = 4·8''$. On the base BC construct an isosceles triangle having the same area.

38. On the base BC of a $\triangle$ABC construct a second triangle equal in area to the first and having its vertex P on a given straight line RS.

39. In a quadrilateral ABCP the lengths AB, BC are given, also the $\angle$ B. Find the locus of the vertex P, if it moves so that the area of the quadrilateral is constant.

40. ABC is a given triangle, and Q a moving point in BA. In CA take a point R such that CR = BQ; and join CQ. Complete the parallelogram QCRP; then find the locus of P.

41. AB is a fixed diameter of a circle. From Q, a point moving on the circumference, QP is drawn parallel to AB and of constant length (say equal to the diameter of the circle). Find the locus of P, and give a proof.

42. ABC is one of a series of right angled triangles described on either side of a given base AB as hypotenuse, and P is the mid-point of AC. If AB is bisected at O, prove that the locus of P, for different positions of C, is the circle on AO as diameter.

43. A $\triangle$ PBC stands on a fixed base BC and is of constant area: find the locus of the intersection of its medians. [See p. 92.]

44. Construct a triangle, having given the base, the altitude, and the length of the median which bisects the base.

45. Take A and B two fixed points 6 cm. apart. Plot (without proof) the locus of a point P that moves so that AP = 2BP.

46. A and B are fixed points on the circumference of a circle, and PQ a diameter varying in position. *Plot* the locus of the intersection of PA and QB.

47. S is a fixed point 2 inches distant from a given straight line MX. Find two points which are $2\frac{1}{4}$ inches distant from S, and also $2\frac{3}{4}$ inches distant from MX.

48. Find a series of points equidistant from a given point S and a given straight line MX. Draw a curve freehand passing through all the points so found.

49. S and S′ are two fixed points. Find a series of points P such that

(i) SP + S′P = constant (say 3·5 inches);

(ii) SP − S′P = constant (say 1·5 inch).

In each case draw a curve freehand passing through all the points so found.

50. A straight line OP revolves about a fixed point O at a uniform rate of 10° per second. Also OP increases in length at the uniform rate of 1 cm. per second. At starting OP was 2 cm. in length. Plot the locus of P during the first 9 seconds.

ANSWERS TO NUMERICAL EXERCISES.

Since the utmost care cannot ensure absolute accuracy in graphical work, results so obtained are likely to be only approximate. The answers here given are those found by calculation, and being true so far as they go, furnish a standard by which the student may test the correctness of his drawing and measurement. Results within one per cent. of those given in the Answers may usually be considered satisfactory.

INTRODUCTION.

II. Measurement of Straight Lines. Pages 4, 5.

6. 1·8″, 3·2″. **7.** 4·5 cm., 8·1 cm.

8. 1·8″, 1·3″, 3·1″. **9.** 8·5 cm., 4·8 cm., 3·7 cm.

10. (i) 3·0″, 1·2″, 1·1″, 0·7″; (ii) 3·0 cm.

13. 2·54 cm. **14.** 0·39″.

15. AB = 3·25″ = 8·2 cm. BC = 0·90″ = 2·3 cm.
CD = 2·23″ = 5·7 cm. DA = 1·60″ = 4·0 cm.
EF = 2·60″ = 6·6 cm. AC = 3·10″ = 7·8 cm.
BD = 2·80″ = 7·1 cm. GH = 1·38″ = 3·5 cm.

III. Circles. Page 8.

11. P is 2·5″ from A and from B. Q is 2″ from A and from B.

12. Two; one on each side of AB. **13.** Two. **14.** Two.

IV. Angles. Pages 13, 15, 16, 19.

1. $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{12}, \frac{5}{12}$. **2.** $\frac{2}{3}, \frac{1}{4}, \frac{1}{8}, \frac{3}{8}, \frac{7}{24}, \frac{13}{20}$.

3. 60°, 45°, 144°, 200°, 300°. **4.** $\frac{1}{2}, \frac{1}{5}, \frac{1}{6}, \frac{9}{16}, \frac{5}{6}$.

5. 360°, 90°, 180°, 270°. **6.** 30°, 150°, 216°.

7. 8 min., 17 min., 1½ min. **8.** 60°. **9.** 7.

10. 42°, 15 hours.

11. ∠APX = 32°, ∠BQX = 60°, ∠CRX = 102°, ∠DSX = 135°;
∠APY = 148°, ∠BQY = 120°, ∠CRY = 78°, ∠DSY = 45°;
∠AOB = 28°, ∠AOC = 70°, ∠AOD = 103°, ∠BOD = 75°;
∠BOC = 43°, ∠COP = 108°, ∠POR = 71°.

14. 95°, 85°, 95°, 85°.

15. CD = 1·8″, ∠ACD = 34°, ∠DCB = 56°, ∠ACB = 90°, ∠ADC = 112°,
∠BDC = 68°.

28. (i) ∠BOC = 37°, ∠COA = 143°, ∠AOD = 37°;
(ii) ∠DOB = 151°, ∠BOC = 29°, ∠COA = 151°;
(iii) ∠BOD = 137°, ∠DOA = 43°, ∠COB = 43°.

GEOMETRY.

V. Direction. Parallels. Pages 21, 22.

1. 45°. 2. 22½°. 9. 44°. 10. (i) 74°; (ii) 106°.

VII. Construction of Triangles. Pages 30-35.

12. 3·5″. 13. ∠A = ∠X = 40°, ∠B = ∠Y = 73°.
16. AB = PQ = 5·3″, AC = PR = 2·3″.
17. AB = XY = 4·3″, BC = YZ = 2·5″.
21. ∠B = ∠Y = 40°, ∠C = ∠Z = 86°.

VIII. Heights and Distances. Pages 39-42.

1. 5 mi. 2. 36 ft 3. 29 ft. 4. 17 ft. 5. 2½ mi.
6. 31 mi. 7. 50·5 m. 8. 7·8 km. 9. 390 yds.
10. 9·9 km. 11. 10 mi. 12. About 30½ mi. N. 25° W.
13. 630 yds 14. 3 km. 15. No, by about 0·1 of a mile.
17. 134 ft 18. 31° 19. 162 m. 20. 505 ft. 21. 159 yds.
22. Nearly 17 mi 23. About 32 mi. 24. 132 ft. 25. 20 ft.

THEORETICAL GEOMETRY.

Exercises. Page 46.

4. 30°; 126°; [illegible]61°; 85°. 11 mm.; 37 mm.
5. 112½°; 155°. 5 hrs. 45 min. 6. 50°; 8 hrs. 40 min.
7. (i) 145°, 35°, 145°. (ii) 55°, 55°. (iii) 86°, 94°.
8. 284°. 9. (i) 67° (ii) 71°. 10. 126°.
11. 36°, 72°, 108°, 144°. 12. 55°.

Exercises. Page 53.

3. ∠EGB = ∠AGH = ∠GHD = ∠CHF = 54°.
∠EGA = ∠BGH = ∠GHC = ∠DHF = 126°.
4. (i) 37°; (ii) 143° 10. (i) 15 secs.; (ii) 30 secs.

Exercises. Page 56.

3. 27°. 4. 92°, 46°. 5. 67°, 6[illegible]°. 6. 30°, 60°, 90°.
7. (i) 36°, 72°, 72°; (ii) 20°, 80°, 80°. 8. 40°. 9. 51°, 111°, 18°
11. (i) 34°; (ii) 107°. 12. 68°. 13. 120°. 14. 36°, 72°, 108°, 144°

Exercises. Page 59.

1. 165°. 3. (i) 5; (ii) 15. 6. (i) 45°; (ii) 36°.
7. (i) 12; (ii) 15.

ANSWERS.

Exercises. Page 85.

[illegible] 54°, 72°, 54°. 25. 36°. 26

[illegible] (i) 16 ; (ii) 45° ; (iii) 1½° per sec.

Exercises. Page 95.

2. 6·80 cm. 3. 2·24″. 4. 0·39. 5. 2·54. [illegible]. 10·6 cm.

Exercises. Page 107.

6. 3·35″. 7. 147 mi. ; 235 km. 1 cm. represents 22 km.

Exercises. Page 111.

6. (i) one ; (ii) two ; (iii) one, right angled ; (iv) one.
7. 346 yds., 693 yds. 8. Results equal. 9 cm.
10. 380 yds. 11. 60°, 120°.
12. (i) one solution ; (ii) two ; (iii) one, right angled ; (iv) impossible.
19. 37 ft. 20. 112 ft. 23. 5·8 cm., 4·2 cm. 24. 7 cm., 8 cm.

Exercises. Page 115.

2. 3·54″. 7. (i) two ; (ii) one. 8. 90°.

Miscellaneous Examples. Page 116.

6. 70°, 82°, 110°, 98°. 10. 78°, 102°. 35. 36°, 168°.

Exercises. Page 122.

1. 2·80 sq. in. 2. 3·50 sq. in. 3. 3·30 sq. in. 4. 3·36 sq. in.
5. 198 sq. m. 6. 42 sq. ft. 7. 10,000 sq. m. 8. 110 sq. ft.
9. 2·6 in. 11. 900 sq. yds. ; 48 yds. ; 4·8 in.
12. 11700 sq. m. 13. 1 cm. = 10 yds. 14. 100 sq. ft.
15. 170 sq. ft. 16. 288 sq. ft. 17. 75 sq. ft.

Exercises. Page 128.

1. (i) 22 sq. cm. ; (ii) 23·6 sq. in. 2. 3·4 sq. in. 4. 1·5″.
5. 1·93″. 6. (i) 180 sq. ft. ; (ii) 8·4 sq. in.
7. (i) 13·44 sq. cm. ; (ii) 15·40 sq. cm. ; (iii) 29·50 sq. cm.
8. 15 sq. cm. 9. 1·3 cm.

Exercises. Page 133.

1. 6 sq. in. 2. 170 sq. ft. 3. 8·4 sq. in. 4. 31·2 sq. cm.

GEOMETRY.

Exercises. Page 136.

1. (i) 25·5 sq. cm. ; (ii) 9·5 sq. in. 2. 693 sq. yds.
3. 12500 sq. m.

Exercises. Page 141.

1. (i) 5 cm. ; (ii) 6·5 cm. ; (iii) 3·[illegible]" 2. (i) 1·[illegible]" ; (ii) [illegible]
3. (i) and (iii). 4. 41 ft. 5. 65 mi. 6. 6·1 km. 7.
8. 48 m. 9. 25 mi. 10. 73 m. 11. 62 ft.

Exercises. Page 142.

12. 7·07 cm. 13. 6·93 cm.

Exercises. Page 144.

1. (i) 20 cm. ; 15 cm. ; (ii) 40 cm. ; 39 cm.
2. 35 cm. ; 12 cm. ; 306 sq. cm.
3. 66 sq. ft. 4. 84 sq. yds. 5. 126 sq. m. 6. 132
7. 180 sq. ft. 8. 306 sq. m. 9. 5·1 cm. nearly.
11. (i) 36 sq. in. ; (ii) 90 sq. ft. ; (iii) 126 sq. cm. ; (iv) 24[illegible]

Exercises. Page 147.

1. 7[illegible] cm.

Exercises. Page 148.

1. 23·90 sq. cm. 2. 8·40 sq. in. 3. 27·52 sq.
4. 129[illegible]0 sq. m. 5. 10·45 sq. m.

Exercises. Page 149.

2. 1·6". 3. 3·1 cm. ; 15·6 sq. cm.

Exercises. Page 150.

3. 10 cm. 9. Side = 2·4". 7. 1·6", 3·0"

Miscellaneous Examples. Page 152.

13. 2·25". 15. 4·3 sq. in. 20. 3·6 cm. ; 30·6 sq.

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